INVERSE SCATTERING AND THE ERNST EQUATION

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1. Introduction

The vacuum Einstein field equations (EFE), in the case of a stationary, axisymmetric spacetime, can be reduced to the Ernst equation, a single scalar equation for a complex-valued function. This equation can then be solved by the *inverse scattering transform* (IST), one of the most important techniques in mathematical physics. We will now briefly describe its history; to avoid obscuring the main ideas, below we will mostly stick to the original evolution equation for which the technique was described, the Korteweg–de Vries equation (the situation for the Ernst equation is somewhat more involved; see [4]). The beauty of the IST and related ideas of soliton theory derives largely from their situation at a wonderful confluence of many branches of mathematics, so along the way we will try to illustrate this as well.

1.1. **Solitons.**

The study of solitons (permanent waves which interact with each other elastically) effectively began with John Scott Russell's observation in 1834 of a boat drawn along a narrow channel by a pair of horses. When the boat suddenly stopped, he noticed a solitary wave was created, and followed it on horseback as it propagated seemingly unchanged down the channel.

In 1895, the **Korteweg–de Vries (KdV) equation** appeared; the form we will use here is

$$K(u) = u_t - 6\underbrace{uu_x}_{\text{nlin.}} + \underbrace{u_{xxx}}_{\text{disp.}} = 0.$$

It is a nonlinear, dispersive wave equation (the -6 is just there for convenience; by rescaling u, x and t we can produce any desired coefficients) derived from making simplifying assumptions in the Navier–Stokes equations. Note that without the dispersive term, shockwaves could form, while without the nonlinear term, dispersion would occur. In a sense it is the competition between the two terms that makes soliton phenomena possible.

There is a classical problem in algebraic geometry (specifically, in the moduli of Riemann surfaces) called the *Schottky problem*. One formulation of this problem asks to characterize the locus Π of period matrices $(\oint_{b_j} \omega_k)$ of compact Riemann surfaces in the *Siegel upper half-space* $\mathfrak{S}_g$, that is, the set of all $g \times g$ symmetric complex matrices with positive-definite imaginary part. The answer, conjectured originally by Novikov and proved by Shiota, involves a generalization of the KdV equation, the Kadomtsev–Petviashvili (KP) equation: it says

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that a point $\tau \in \mathfrak{S}_g$ is a period matrix if and only if its associated Riemann theta function

$$\Theta(z,\tau) = \sum_{\vec{m}} \exp\left(2\pi i \left(\frac{1}{2} m^\top \tau m + m^\top z\right)\right)$$

(or rather a certain function derived from it) satisfies the KP equations ("Hirota bilinear relations").

1.2. **GGKM and inverse scattering.** Significant motivation for progress on the KdV equation came from the 1955 Fermi–Pasta–Ulam–Tsingou problem. In 1967, two years after computer simulations revealed some intriguing properties of soliton solutions of KdV, there appeared a paper [10] of Gardner, Greene, Kruskal and Miura (GGKM) detailing a solution method for the KdV equation. The key idea is to introduce a solution $u(x,t)$ of the KdV equation as a potential in the time-independent Schrödinger equation of quantum mechanics. They then show that various associated quantities (the "scattering data") are either invariant or evolve in a very manageable way as u evolves according to KdV. From the scattering data, one can then recover the potential $u(x,t)$ at any time $t = t_0$ by solving an integral equation.

In 1968, Lax introduced the notion of *Lax pairs*, generalizing the method of GGKM, which we will say a bit more about below. In 1973, Ablowitz, Kaup, Newell and Segur managed to solve the sine-Gordon equation similarly. In 1974, the paper [3] appeared, in which (by analogy with the method of Fourier transforms for linear problems) the term "inverse scattering transform" was coined.

Miura discovered a way of converting solutions of the *modified KdV equation*

$$M(v) = v_t - 6v^2 v_x + v_{xxx} = 0$$

to solutions u of KdV: simply substitute $u = v^2 + v_x$. Specifically,

$$K(u) = \left(2v + \frac{\partial}{\partial x}\right) M(v).$$

This turned out to be quite important. Firstly, it serves as the basis of a proof that KdV had an infinite number of conservation laws, which had previously been conjectured. Secondly, it motivated the development of the IST: the Miura substitution may be viewed as a Riccati equation for v in terms of u. This leads quickly to the Schrödinger equation

$$v_{xx} - (u - \lambda)v = 0.$$

1.3. **Integrability.** More generally, one can wonder whether it is possible to identify, in some sense, which of these nonlinear PDEs should be considered "completely integrable". In the classical setting of Hamiltonian systems with finitely many degrees of freedom, it is quite clear what one might mean by this, and there is a well-developed theory of (Liouville) integrability.

In contrast, for nonlinear PDEs, we have in some sense moved from mechanics to field theory (the latter involving things like the motion of a string). The conditions under which nonlinear PDEs can be solved are still not well-understood, even in one dimension; indeed an entire book [18] has been written on this subject. Nonetheless, there appear to so far be three main candidates for a definition of integrability: the existence of a Lax formulation (i.e. expressibility as a compatibility condition for an overdetermined linear system), the Painlevé property (a condition on the singularities of a differential equation), and finally

expressibility as a dimensional reduction of the anti-self-dual Yang–Mills equations. None of these are completely satisfactory. The final one seems to be the most promising, since it at least includes essentially all the classical examples in two and three dimensions, but as Hitchin put it in [12], the KP equation must be "ruthlessly hacked and stretched to fit the Procrustean bed of self-duality".

1.4. **Other perspectives.** Some ideas closely related to the inverse scattering method are Bäcklund transformations and Riemann–Hilbert (RH) problems. In [1, §7.7, p. 613], for example, it is mentioned that the inverse scattering problem associated with the time-independent Schrödinger equation, as well as the initial value problem for the KdV equation, can in many cases be reduced to a (vector) Riemann–Hilbert problem. In general, vector RH problems cannot be solved in closed form; solution can be given in terms of linear integral equations of Fredholm type. Reconstruction of functions via the classical Fourier transform and the Radon transform can also be viewed as RH problems.

2. Inverse scattering method

2.1. **Scattering problem for the Schrödinger equation.** We now describe the inverse scattering method in more detail. Following [7, Ch. 2], we begin with the scattering problem for the Schrödinger equation on the infinite line,

$$v_{xx} - (u(x) - \lambda)v = 0, \qquad x \in \mathbf{R}.$$

We fix a potential $u(x)$ that satisfies the integrability condition

$$(*) \qquad \int_{-\infty}^{\infty} |u(x)||x|^k \, dx < \infty, \qquad k = 0, 1, 2$$

and search for values of λ (called the **eigenvalues** of the equation) for which there exist solutions $v(x)$ that are bounded as $|x| \to \infty$. We will be fairly liberal with regularity assumptions, as our goal is not to find the most general conditions under which the analysis can be made to work. There are a finite number of discrete simple eigenvalues $\lambda = \lambda_n = -k_n^2$ with k_n positive real numbers, which are such that the corresponding eigenfunctions $\psi_n(x)$ belong to $L^2(\mathbf{R})$. We take these eigenfunctions to be normalized by

$$\|\psi_n\|_{L^2(\mathbf{R})} = 1, \qquad \psi_n(x) > 0 \text{ for } x \to +\infty.$$

Due to the decay condition on u, one can see from the Schrödinger equation the following asymptotic behaviour of the eigenfunctions:

$$\psi_n(x) \sim c_n e^{-k_n x}, \qquad x \to \infty$$
$$\psi_n(x) \sim \tilde{c}_n e^{k_n x}, \qquad x \to -\infty.$$

In particular we can define the **normalization coefficients** by

$$c_n = \lim_{x \to \infty} e^{k_n x} \psi_n(x).$$

There also exist solutions of the Schrödinger equation which are bounded for $|x| \to \infty$, for

$$\lambda = +k^2, \qquad \forall k \in \mathbf{R}, \qquad k \neq 0.$$

These solutions, which we denote $\psi_k(x)$, behave for $x \to \pm\infty$ as a linear combination of e^{-ikx} and e^{ikx}. Similarly to before, we impose the following normalization:

$$\psi_k(x) \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{for } x \to +\infty \\ a(k)e^{-ikx} & \text{for } x \to -\infty. \end{cases}$$

We call $a(k)$ and $b(k)$ the **reflection** (resp. **transmission**) **coefficient**. The collection of eigenvalues, together with the coefficients c_n, $a(k)$ and $b(k)$ make up the **scattering data** associated to the potential $u(x)$. We can then ask whether it is possible to recover the potential $u(x)$ from this scattering data; this is the **inverse scattering problem**.

As it turns out, it is indeed possible. The outline is as follows: we define a function $B(\zeta)$ by

$$B(\zeta) = \sum_{n=1}^{N} c_n^2 e^{-k_n \zeta} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k) e^{ik\zeta}\, dk,$$

where N is the number of discrete eigenvalues (first term is absent if $N = 0$). We then let $K(x, y)$ be a solution to the so-called **Gelfand–Levitan–Marchenko integral equation**,

$$K(x, y) + B(x + y) + \int_x^{\infty} B(z + y)K(x, z)\, dz = 0, \qquad y > x.$$

Then the potential is recovered as

$$u(x) = -2\frac{d}{dx}K(x, x).$$

Our original goal, of course, was to solve the initial-value problem for the KdV equation. However, the above brings us very close: since we can now recover the potential from the scattering data, it now remains only to understand how this scattering data must evolve in order that the recovered $u(x, t)$ solve the KdV equation. As it turns out, the evolution is quite tractable.

2.2. **Isospectrality.** The first fundamental fact about the time evolution is *isospectrality* (i.e. the Schrödinger eigenvalues are invariant in time). We start with the following:

Theorem 1. *If there exists a differential operator M such that $L_t = [M, L]$ where $L = -\partial_x^2 + u(x, t)$ then the eigenvalues of L do not depend on t.*

Proof. Consider the eigenvalue problem $Lv = \lambda v$. Differentiating gives

$$L_t v + L v_t = \lambda_t v + \lambda v_t.$$

Note that $MLv = \lambda Mv$ and use $L_t = [M, L]$ to find $(L - \lambda)(v_t + Mv) = \lambda_t v$. Take the inner product of this equation with v, and use the fact that L is self adjoint:

$$\lambda_t \|v\|^2 = \langle v, (L - \lambda)(v_t + Mv)\rangle = \langle (L - \lambda)v, v_t + Mv\rangle = 0$$

therefore $\lambda_t = 0$. ∎

Note that since we used the L^2 inner product above, the argument only applies in our case when the eigenvector v lies in $L^2(\mathbf{R})$, i.e. for the discrete eigenvalues. However, it can also be shown for continuous eigenvalues.

We will see that such an M does in fact exist. Thus let us describe the basic idea behind the Lax approach; see for example [2] for further details. Consider two operators L and M, where L is the operator of the spectral problem while M is the operator of an associated time evolution equation:

$$Lv = \lambda v, \qquad v_t = Mv.$$

Differentiating the first with respect to time, assuming $\lambda_t = 0$, we obtain $L_t v + L v_t = \lambda v_t$. Substituting in the second equation gives the *Lax equation*,

$$L_t + [L, M] = 0.$$

This equation is required for the system to be compatible; it contains a nonlinear evolution equation if L and M are correctly chosen. For example for KdV,

$$L = -\partial_x^2 + u, \qquad M = -4\partial_x^3 + 3(u\partial_x + \partial_x(u\bullet)).$$

Proposition 2. *$u(x,t)$ solves KdV if and only if L and M satisfy $L_t + [L, M] = 0$.*

Proof. Note that $L_t = u_t$, since the coefficient of $-\partial_x^2$ is independent of time. For the equation to hold, then, all terms with positive powers of ∂ in $[M, L]$ must cancel. Some simple but tedious algebraic manipulations with differential operators completes the proof. ■

To summarize, we state the following.

Theorem 3. *Let $u(x,t)$ be a solution of the KdV equation, satisfying the integrability condition (*) and which is such that for $p = 1, 2, 3$,*

$$\frac{\partial^p u(x,t)}{\partial x^p}$$

is bounded for $|x| \to \infty$. Then the eigenvalues of the Schrödinger equation are invariant in time.

2.3. **Evolution of normalization coefficients and reflection coefficients.** One can show (see for example the beginning of [3]) that the normalization and reflection coefficients evolve according to some simple linear ODEs, which are easily integrated explicitly. Thus in theory we are done; in practice however the Gelfand–Levitan–Marchenko equation can only be solved explicitly in simple cases.

3. Ernst equation

To make precise what is meant by a stationary axisymmetric spacetime, we introduce the concept of a Killing vector. A **Killing vector** (or **Killing field**) is just a vector field on the spacetime whose flow is an isometry. We say the spacetime is **stationary** if it admits a timelike Killing vector; this means that with respect to some coordinate system, the metric components are all independent of the time coordinate. Similarly we define what it means for a spacetime to be **axisymmetric**. In [14, Ch. 2], the full details are given of the reduction of the vacuum EFE under these symmetry assumptions to the **(elliptic) Ernst equation**, which reads

$$(\mathcal{E} + \overline{\mathcal{E}})(\mathcal{E}_{tt} + \rho^{-1}\mathcal{E}_\rho + \mathcal{E}_{\rho\rho}) = 2(\mathcal{E}_\rho^2 + \mathcal{E}_t^2),$$

where $\mathcal{E}$ is a complex-valued function called the **Ernst potential**. From the Ernst potential, the components of the metric may be obtained by quadratures. It was established in the paper [4] that the Ernst equation is in fact integrable via the IST.

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