MATH 595/597 ASSIGNMENT 4

DUE WEDNESDAY DECEMBER 3

1. Prove that $L^1(I,H)^* \cong L^\infty(I,H)$, where $I \subset \mathbb{R}$ is an open interval, and H is a separable Hilbert space.
2. Consider the Cauchy problem for the *Euler equations*

$$\partial_t u + \mathbb{P}\mathrm{div}(u \otimes u) = 0,$$

where $\mathbb{P}$ is the Leray projector. The initial data g is divergence free and is in $H^s(\mathbb{T}^n)^n$ for some $s > \frac{n}{2} + 1$. We want to prove local well posedness in H^s, by approximating the Euler equations by the Navier-Stokes equations. The following is a rough roadmap and some details may have to be tweaked to work.

a) Prove uniqueness in class $\mathscr{C}([0,T], H^s)$.

b) Prove that there exists $T > 0$, depending only on the H^s-norm of g, and in particular independent of $\varepsilon > 0$, such that the Navier-Stokes equations

$$\partial_t u + \mathbb{P}\mathrm{div}(u \otimes u) = \varepsilon \Delta u,$$

admit a solution $u_\varepsilon \in \mathscr{C}([0,T], H^s)$, with uniformly bounded norms.

c) By compactness arguments, construct a solution of the Euler equations satisfying $u \in L^\infty((0,T), H^s) \cap \mathscr{C}([0,T], H^\sigma)$ for any $\sigma < s$.

d) Show that $u : [0,T] \to H^s$ is continuous when H^s is equipped with its weak topology. Then prove that indeed $u \in \mathscr{C}([0,T], H^s)$ by showing that the norm $\|u(t)\|_{H^s}$ varies continuously in t.

3. Consider the Navier-Stokes initial value problem

$$\partial_t u + \mathbb{P}\mathrm{div}(u \otimes u) = \Delta u + \mathbb{P}f, \qquad (*)$$

with (divergence free) initial data $g \in H^s(\mathbb{T}^n)^n$ for some $s > \frac{n}{2}$, and the forcing term $f \in L^2_{\mathrm{loc}}((0,\infty), L^2(\mathbb{T}^n)^n)$. We want to study the approximation of $(*)$ by the so-called *Hopf-Galerkin method*. To this end, let $P_m : L^2(\mathbb{T}^n)^n \to \Sigma_m$ be the Fourier truncation operator defined by $\widehat{P_m u}(\xi) = \hat{u}(\xi)$ for $\xi \in [-m, \dots m]^n$ and $\widehat{P_m u}(\xi) = 0$ otherwise. The space Σ_m is the range of P_m, which is a finite dimensional subspace of $L^2(\mathbb{T}^n)^n$ consisting of trigonometric polynomials. We consider the initial value problem

$$\partial_t u_m + \mathbb{P}P_m \mathrm{div}(u_m \otimes u_m) = \Delta u_m + \mathbb{P}P_m f, \qquad (*)$$

for $u_m : [0,\infty) \to \Sigma_m$, with initial data $u_m(0) = P_m g$. Note that since Σ_m is finite dimensional, we can approximately solve $(*)$ on a computer, by using standard ODE discretization methods in time.

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In the following, you can assume more regularity or decay on f if necessary. For each result, try to identify the weakest conditions on f possible.

a) Show that $(*)$ has a global solution $u_m \in \mathscr{C}([0,\infty),\Sigma_m)$, satisfying uniform bounds in $L^\infty((0,T),L^2)$ and $L^2((0,T),H^1)$ for each $T>0$. What do we need to assume on f in order to have the same bounds with $T=\infty$?

b) Show that a subsequence of $\{u_m\}$ converges to a weak solution of the Navier-Stokes problem $(*)$. Try to derive the strongest conclusions with regard to the mode of convergence.

c) For $n=2$, prove that the whole sequence $\{u_m\}$ converges to the unique strong solution of the Navier-Stokes problem $(*)$. Try to derive the strongest conclusions with regard to the mode of convergence.