MATH 595/597 ASSIGNMENT 2

DUE WEDNESDAY OCTOBER 22

1. Consider the initial value problem
$$\partial_t u = \Delta u + f(u), \qquad u|_{\{t=0\}} = g,$$
where $f : H^s(\mathbb{T}^n) \to H^{s-1}(\mathbb{T}^n)$ is locally Lipschitz, and $g \in H^s(\mathbb{T}^n)$, for some constant $s \geq 1$. We know that there exists a unique maximal mild solution $u_g \in C([0, T_g), H^s(\mathbb{T}^n))$, with the maximal time of existence $0 < T_g \leq \infty$ possibly depending on the initial datum $g \in H^s(\mathbb{T}^n)$. For $t \geq 0$ fixed, let $\Omega_t = \{g \in H^s(\mathbb{T}^n) : T_g > t\}$, and define the flow map $\Phi_t : \Omega_t \to H^s(\mathbb{T}^n)$ by $\Phi_t g = u_g(t)$. Prove that the solution depends on the initial data continuously in the following sense. For any $g \in H^s(\mathbb{T}^n)$ and $t \in [0, T_g)$, there exists $\delta > 0$ such that $B_\delta(g) = \{h \in H^s(\mathbb{T}^n) : \|h - g\|_s < \delta\} \subset \Omega_t$, and that $\Phi_t : B_\delta(g) \to H^s(\mathbb{T}^n)$ is Lipschitz continuous.
2. Consider the following nonlinear reaction-diffusion equation
$$\partial_t u = \Delta u + \kappa e^{\kappa u},$$
with initial datum $g \in H^s(\mathbb{T}^n)$, where $\kappa \in \mathbb{R}$ and $s > \frac{n}{2}$ are constants. We assume that g is positive everywhere. Prove the following, and in the case of (d), provide upper and lower bounds for the blow-up time T.
(a) The problem has a unique maximal mild solution $u \in C([0, T), H^s(\mathbb{T}^n))$.
(b) The mild solution is in fact smooth in $\mathbb{T}^n \times (0, T)$.
(c) We have $u > 0$ pointwise in $\mathbb{T}^n \times (0, T)$.
(d) If $\kappa > 0$, then the solution blows up in a finite time, i.e., $T < \infty$.
(e) If $\kappa \leq 0$, then the solution is global in time, meaning that $T = \infty$.
(f) If $\kappa < 0$, then $u(t)$ decays to 0 as $t \to \infty$.
3. Consider the Allen-Cahn equation
$$\partial_t u = \Delta u + u - u^3,$$
with initial datum $g \in H^s(\mathbb{T}^n)$ for some $s > \frac{n}{2}$. Prove the following.
(a) The problem has a unique maximal mild solution $u \in C([0, T), H^s(\mathbb{T}^n))$.
(b) The mild solution is in fact smooth in $\mathbb{T}^n \times (0, T)$.
(c) The solution is global in time.

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