MATH 595/597 ASSIGNMENT 1

DUE FRIDAY OCTOBER 3

1. Prove the following.
   (a) If $a \in \ell^2(\mathbb{Z}^n)$ then there exists $g \in L^2(\mathbb{T}^n)$ such that
   $$g = \lim_{m\to\infty} \sum_{k\in Q_m} a_k e_k, \qquad (*)$$
   with the convergence in $L^2(\mathbb{T}^n)$, where $Q_m = \{-m, \ldots, m\}^n$ and $e_k(x) = e^{ik\cdot x}$ for $k \in \mathbb{Z}^n$.
   (b) Conversely, if $g \in L^2(\mathbb{T}^n)$ and
   $$\hat{g}(k) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} g(x)e^{-ik\cdot x}\,\mathrm{d}x = \frac{1}{(2\pi)^n}\langle g, e_k\rangle, \qquad k \in \mathbb{Z}^n,$$
   then we have $\hat{g} \in \ell^2(\mathbb{Z}^n)$ and $(*)$ holds with $a_k = \hat{g}(k)$.
2. Consider the *hyperdissipative heat equation*
   $$u'(t) = -|D|^\theta u(t), \qquad \text{for } t > 0, \qquad (**)$$
   where $\theta > 0$ and the operator $|D|^\theta$ is given in Fourier space by $\widehat{|D|^\theta f}(k) = |k|^\theta \hat{f}(k)$. Note that $\theta = 2$ corresponds to the standard heat equation. Let
   $$u(t) = e^{-t|D|^\theta} g = \sum_{k\in\mathbb{Z}^n} e^{-|k|^\theta t}\hat{g}(k)e_k, \qquad t \geq 0, \qquad (\dagger)$$
   where $g \in L^2(\mathbb{T}^n)$. Prove the following.
   (a) Let $u, v \in C([0,T), L^2(\mathbb{T}^n))$ with $T > 0$ be two functions satisfying $(**)$ as an equality in $L^2(\mathbb{T}^n)$ for all $0 < t < T$, and let $u(0) = v(0)$. Then $u = v$ on $[0,T)$.
   (b) For any $s \geq 0$ and $t > 0$, the propagator $e^{-t|D|^\theta} : L^2(\mathbb{T}^n) \to H^s(\mathbb{T}^n)$ is bounded, with
   $$|e^{-t|D|^\theta} g|_s \leq Ct^{-s/\theta}\|g\|, \qquad g \in L^2(\mathbb{T}^n),$$
   where C is a constant depending only on s and θ.
   (c) The function $u : (0,\infty) \to H^s(\mathbb{T}^n)$ given in $(\dagger)$ satisfies $u \in C^\infty((0,\infty), H^s(\mathbb{T}^n))$ for any $s \geq 0$.
   (d) If $g \in H^\sigma(\mathbb{T}^n)$ for some $\sigma \geq 0$, then $u(t) \to g$ in $H^\sigma(\mathbb{T}^n)$ as $t \searrow 0$. Moreover, u satisfies $(**)$ as an equality in $H^s(\mathbb{T}^n)$ for $t > 0$ and $s \geq 0$.
   (e) The quantity $u(t)(x)$, considered as a function of $(x,t) \in \mathbb{T}^n \times (0,\infty)$, is smooth.

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3. Consider the *inhomogeneous hyperdissipative heat equation*

$$u'(t) = -|D|^\theta u(t) + f(t), \qquad \text{for} \quad 0 < t < T, \tag{‡}$$

where $\theta > 0$ and $f \in C((0,T), H^\sigma(\mathbb{T}^n))$ for some $\sigma \geq 0$ and $0 < T \leq \infty$. We will impose the initial condition $\lim_{t \searrow 0} u(t) = g$ in $L^2(\mathbb{T}^n)$, and will understand that u' is the derivative of u considered as a function $u : (0,T) \to H^s(\mathbb{T}^n)$ for some suitable $s \geq 0$. In other words, we look for a *strong H^s-solution* of $\partial_t u = -|D|^\theta u + f$.

Prove the following.

(a) Let $u \in C([0,T), L^2(\mathbb{T}^n))$ and $f \in C((0,T), L^2(\mathbb{T}^n))$ satisfy (‡) and $u(0) = g$. In particular, for each $t \in (0,T)$, $u'(t)$ and $|D|^\theta u(t)$ both exist in $L^2(\mathbb{T}^n)$. We also assume that

$$\lim_{\varepsilon \searrow 0} \int_\varepsilon^a \|f(t)\| \,\mathrm{d}t < \infty, \qquad \text{for some} \quad 0 < a < T.$$

Then we have

$$u(t) = e^{-t|D|^\theta} g + \lim_{\varepsilon \searrow 0} \int_\varepsilon^t e^{(\tau - t)|D|^\theta} f(\tau) \,\mathrm{d}\tau, \qquad (0 < t < T),$$

where the limit $\varepsilon \to 0$ can be replaced by the evaluation $\varepsilon = 0$ if $f \in C([0,T), L^2(\mathbb{T}^n))$. In particular, (‡) has uniqueness in the considered class.

(b) Let $g \in H^\alpha(\mathbb{T}^n)$ and let $f \in C((0,T), H^\sigma(\mathbb{T}^n))$, with $0 \leq \alpha \leq \sigma$. Also let

$$\lim_{\varepsilon \searrow 0} \int_\varepsilon^a \|f(t)\|_\sigma \,\mathrm{d}t < \infty, \qquad \text{for some} \quad 0 < a < T.$$

Then the function

$$u(t) = e^{-t|D|^\theta} g + \lim_{\varepsilon \searrow 0} \int_\varepsilon^{t-\varepsilon} e^{(\tau - t)|D|^\theta} f(\tau) \,\mathrm{d}\tau, \qquad (0 < t < T),$$

is in $C((0,T), H^s(\mathbb{T}^n))$ for all $\alpha \leq s < \sigma + \theta$. Furthermore, we have $\lim_{t \searrow 0} u(t) = g$ in H^α, and u is a strong H^s-solution of (‡) for $s < \sigma$.