The Navier-Stokes Equations on $\mathbb{T}^2$ with Stochastic Forcing

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Abstract. These notes results from a class project in MATH 595, a topics course on the analysis of the Navier-Stokes equations, at McGill University during the Fall semester 2014. They are heavily based on a set of notes written by Martin Hairer [1]. Their goal is to give a proof of the existence of a unique maximal solution to the Navier-Stokes equations on the two-dimensional torus under a stochastic forcing and to show that such a solution is global in time. The approach is incremental: linear equation without stochastic force will be studied first with the help of semigroups, linear stochastic PDEs (SPDEs) and then semilinear SPDEs will be introduced in a second time. Finally, the Navier-Stokes equations will be deal with as an example of a semilinear SPDE by using the Leray projector.

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1. Motivation

The goal of these notes is to show the following:

Theorem 1 (Existence and uniqueness of a global solution). *The stochastic Navier-Stokes equations*

$$\begin{cases} du = \Delta u \, dt - (u \cdot \nabla) u \, dt - \nabla p \, dt + Q \, dW_t, \\ \operatorname{div} u = 0 \end{cases}$$

admit a unique global solution on the torus $\mathbb{T}^2$.

To solve the stochastic Navier-Stokes equations, we first have to rewrite them into a more general form, i.e. into a stochastic semilinear system of equations. With the help of some semigroup theory and the notion of a stochastic convolution, we will prove the existence and uniqueness of a maximal solution to equations of this form under a few additional weak assumptions (that will be satisfy for the Navier-Stokes equations). We will then prove that this maximal solution is global in time with an *a priori* energy estimate using negative order Sobolev spaces with the clever introduction of a vorticity to the system.

2. Semigroup

This section deals with linear equation without stochastic forcing, i.e. with equations of the form

$$\partial_t x = Lx, \qquad x(0) = x_0 \in \mathcal{B}$$

for some Banach space $\mathcal{B}$ and a linear operator L. We will see that semigroups are the unique solution to such a linear equation. Semigroups are necessary to study more complicated cases with stochastic forcing and/or non-linearity.

Definition 1 (Semigroup). *Let $\mathcal{B}$ be a Banach space. A family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on $\mathcal{B}$ satisfying*

$$\text{(A)} \qquad S(0) = \text{Identity}$$

$$\text{(B)} \qquad S(t) \circ S(s) = S(t+s) \text{ for any } s,\, t \geq 0$$

is called a semigroup on $\mathcal{B}$ and is denoted by $S(t)$. If

$$\lim_{t\to 0} \|S(t)x - x\| = 0 \text{ for every } x \in \mathcal{B},$$

then the semigroup $S(t)$ is said to be strongly continuous.

A more restrictive but also more useful property is the following.

Definition 2 (Analytic semigroup). *A semigroup $S(t)$ is called analytic if there exists an angle $\theta > 0$ such that the map $t \mapsto S(t)$ has an analytic extension on the set $\{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}$, the extension satisfies the semigroup property on the set and the map $t \mapsto S(e^{i\phi}t)$ is a strongly continuous whenever $|\phi| < \theta$.*

As with any function of time, it is interesting to ask ourself if we can make sense of some kind of derivative. The following definition shows that we can in a certain general sense. In probability, an infinitesimal generator is often seen as a derivative at $t = 0$ of a stochastic process.

DEFINITION 3 (Infinitesimal generator). *The infinitesimal generator L of a strongly continuous semigroup $S(t)$ on the Banach space $\mathcal{B}$ is the operator given by*

$$Lx = \lim_{t \to 0} \frac{S(t)x - x}{t}.$$

The set of $x \in \mathcal{B}$ such that this limit exists is called the domain of L and is denoted by $\mathcal{D}(L)$.

The strength and importance of semigroups resided in the next proposition that shows that if L is the infinitesimal generator of a strongly continuous semigroup $S(t)$, then $S(t)x_0$ is a solution to the linear equation $\partial_t x = Lx$.

THEOREM 2 (Existence of a solution). *The domain $\mathcal{D}(L)$ of L is dense in $\mathcal{B}$ and invariant under the action of the semigroup $S(t)$. For every $t \geq 0$ and every $x \in \mathcal{D}(L)$, we have*

$$\partial_t S(t)x = LS(t)x.$$

Furthermore, $S(t)$ commutes with its infinitesimal generator L, i.e. $LS(t)x = S(t)Lx$ for every $t \geq 0$ and every $x \in \mathcal{D}(L)$.

THEOREM 3 (Uniqueness of the solution). *If $x : [0,1] \to \mathcal{D}(L)$ satisfies $\partial_t x = Lx$ on $[0,1]$, then $x_t = S(t)X_0$, i.e. the solution is unique.*

EXAMPLE 1 (Heat semigroup).

The next result will allows us to have an invariant on the interpolation spaces of the infinitesimal generator of an analytic semigroup under a perturbation of the infinitesimal generator. An important application is that we don't always have to study a complicated generator, knowing well an 'easier' version is enough if we can perturbed it to the more complicated one.

THEOREM 4 (Pertubation of the generator of an analytic semigroup).

The proof of the above theorem would require a good amount of spectral theory and is not related to our goals. We thus omit the proof from those notes.

3. Interpolation spaces

While we will only need Sobolev spaces to prove the existence of a unique global solution to the Navier-Stokes equation with stochastic forcing in 2-dimension, we will still need a few properties that the Sobolev spaces inherited from the general theory of interpolation spaces. We thus develop those tools in this section.

DEFINITION 4 (Interpolation space). *If $S(t)$ is an analytic semigroup on the Banach space $\mathcal{B}$ and $\alpha > 0$, then the domain of $(-L)^{\alpha}$ under the norm $\|x\|_{\mathcal{B}_\alpha} = \|(-L)^{\alpha}x\|$ is called the interpolation space $\mathcal{B}_\alpha$ of the infintisimal generator L.*
To get negative order interpolation spaces, we define $\mathcal{B}_{-\alpha}$ for $\alpha > 0$ as the completion of $\mathcal{B}$ under the norm $\|x\|_{\mathcal{B}_{-\alpha}} = \|(-L)^{-\alpha}x\|$ where the negative order operator $(-L)^{-\alpha}$ is given by $(-L)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha - a} S(t)dt$.

Interpolation spaces measure the smoothness of their members in a very general way. Because they are defined as $\mathcal{D}((-L)^{\alpha})$, i.e. as the domain of the fractional order α of an infinitesimal generator, we get that $\mathcal{B}_\alpha \subset \mathcal{B}_\beta$ whenever $\alpha \leq \beta$. An element x of $\mathcal{B}_\beta$ is smoother then an element y of $\mathcal{B}_\alpha$ in the sense that we can apply more times L to x that we can to y. This is in direct analogy with the more familiar case were L is a differential operator.

For the Navier-Stokes equation, we will need to know more about the interpolation spaces of the Laplacian. However, the following example shows that we already know what those interpolation spaces are.

EXAMPLE 2 (Interpolation spaces of the Laplacian Δ).

The next lemma will be useful to our final proof in the last section.

LEMMA 1 (Interpolation spaces of polynomial differential operators).

As for the heat semigroup that we saw in class, semigroups have a smoothing effect in general. The next theorem goes in this direction.

THEOREM 5 (Smoothing effect of a semigroup). *The semigroup $S(t)$ maps $\mathcal{B}$ into $\mathcal{B}_\alpha$ for every $t > 0$ and every $\alpha > 0$. Furthermore, for an arbitrary α, there exists a constant C_α depending only on α such that*

$$\|(-L)^{\alpha}S(t)x\| \leq \frac{C_\alpha}{t^\alpha} \qquad \text{for every } t \in (0,1] \text{ and every } x \in \mathcal{B}.$$

We saw in class that the heat semigroup $e^{t\Delta}$ satisfies a second property, namely that $e^{t\Delta}x \to x$ as $t \to 0$. This is true for a general analytic semigroup $S(t)$. Interpolation spaces allows us to bound the speed of this convergence as follows:

THEOREM 6 (Decay of an analytic semigroup as $t \to 0$). *If $S(t)$ is an analytic semigroup on a Banach space $\mathcal{B}$ with infinitesimal generator L, then for an arbitrary α in $(0,1)$ there exists a constant C_α such that*

$$\|S(t)x - x\| \leq C_\alpha t^{\alpha} \|x\|_{\mathcal{B}_\alpha} \qquad \text{for every } t \in (0,1] \text{ and every } x \in \mathcal{B}_\alpha.$$

We conclude this section with the following theorem.

THEOREM 7 (Invariance of the interpolation spaces under a translation of the generator by a bounded operator). *Let $\mathcal{B}_\gamma^0$ be the interpolation spaces of an infinitesimal generator L_0 of an analytic semigroup on the Banach* *space $\mathcal{B}$. Let $\mathcal{B}_\gamma$ be the interpolation spaces associated to the infinitesimal generator $L_0 + B$, where $B : \mathcal{B}^0_\alpha \to \mathcal{B}$ (for some α in $[0,1)$) is a bounded operator. For every γ in $[0,1]$, we have the equality $\mathcal{B}_\gamma = \mathcal{B}^0_\gamma$.*

Again, the proof is omit to stay focused on the Navier-Stokes equations.

4. Linear SPDEs

In order to solve the stochastic Navier-Stokes equations, we have to introduce a few notions from the theory of stochastic partial differential equations (SPDEs). We are mostly interested in semilinear equations, but before being able to understand them, we first have to spend some time on the linear case.

First, we make clear what we mean by a solution to a linear SPDE. We will then show that such a solution has almost surely continuous sample paths and that it is regular in space and in time. Finally, we will argue that some solutions are global in time.

4.1. Linear SPDEs' solutions.

Definition 5 (Stochastic linear equation). *By a stochastic linear equation on a Banach space $\mathcal{B}$, we mean an equation of the form*

$$(\dagger) \qquad \begin{cases} du = Lu\,dt + Q\,dW_t, \\ u(0) = u_0 \in \mathcal{B} \end{cases}$$

where L is the generator of a strongly continuous semigroup (a $\mathcal{C}_0$-semigroup) on $\mathcal{B}$, $Q : \mathcal{H} \to \mathcal{B}$ is a bounded linear operator and W is a cylindrical Wiener process on the Hilbert space $\mathcal{H}$.

To make sense of what a solution to $(\dagger)$ is, we begin with a first notion. In the next definition, $\mathcal{D}(L^*)$ is the domain of L^*, the adjoint operator of the generator L.

Definition 6 (Weak solution). *A $\mathcal{B}$-valued process $u(t)$ is said to be a weak solution to $(\dagger)$ if the following two conditions are met:*

$$\text{(A)} \qquad \text{for every } t > 0,\ \int_0^t \|u(s)\|_{\mathcal{B}}\,ds < \infty \text{ almost surely}$$

$$\text{(B)} \qquad \langle \ell, u(t)\rangle = \langle \ell, u_0\rangle + \int_0^t \langle L^*\ell, u(s)\rangle + \int_0^t \langle Q^*\ell, dW_s\rangle.$$

The second condition must holds almost surely for every $\ell \in \mathcal{D}(L^*)$.

Weak solutions won't be enough for our application to the stochastic Navier-Stokes equations however. We need the more general notion of a mild solution to (†). To this end, let $S(t)$ be the strongly continuous semigroup generated by L.

Definition 7 (Mild solution). *A $\mathcal{B}$-valued process $u(t)$ is said to be a mild solution to (†) if*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)Q\,dW_s$$

almost surely for every $t > 0$.

As said before, mild solutions are more general than weak solutions. Specifically, a weak solution is always mild. Conversely, a mild solution always satisfies the condition (B) of the definition of a weak solution. If we add the hypothesis that for every $t > 0$ the mild solution is almost surely integrable (condition (A)), then it is also a weak solution.

4.2. Regularity of the solutions.

We want our solutions to have three regularity properties. Namely that they have almost surely continuous sample paths, that they are regular in space and finally that they are regular in time. The following three theorems and their proofs give these results.

Theorem 8 (Continuity of the sample paths). *Let $\mathcal{H}$ be a separable Hilbert Space, let L be the infinitesimal generator of a strongly continuous semigroup $S(t)$ on $\mathcal{H}$, let $Q : \mathcal{H} \to \mathcal{H}$ be a bounded operator and let W be a cylindrical Wiener process on $\mathcal{H}$. If*

$$\text{(A)} \qquad \|S(t)Q\|_{HS} < \infty \text{ for every } t > 0 \qquad \text{and}$$

$$\text{(B)} \qquad \int_0^1 t^{-2\alpha}\|S(t)Q\|^2_{HS}dt < \infty \text{ for some } \alpha \in (0, \frac{1}{2})$$

then a solution u to (†) has almost surely continuous sample paths in $\mathcal{H}$.

Proof. Let $y(t) = \int_0^t (t-s)^{-\alpha}S(t-s)Q\,dW_s$. We will show that the a map taking $y(t)$ to the solution $u(t)$ of (†) is a map from $L^p([0,T],\mathcal{H})$ to $\mathcal{C}([0,T],\mathcal{H})$ for any $T > 0$, hence proving that the sample paths are almost surely continuous.
Using hypothesis (A) with $t+s$, together with the semigroup property of $S(t+s)$, we get that $\|S(t+s)Q\|_{HS} \leq \|S(s)\|\|S(t)Q\|_{HS}$. Using this with assumption (B) now yields that $\int_0^T t^{-2\alpha}\|S(t)Q\|^2_{HS}dt < \infty$ for any finite

time $T > 0$. This gives us that

$$\begin{aligned}\mathbb{E}\left[\|y(t)\|^2\right] &= \int_0^t (t-s)^{-2\alpha}\|S(t-s)Q\|_{HS}^2 ds \\ &= \int_0^t s^{-2\alpha}\|S(s)Q\|_{HS}^2 ds < \infty.\end{aligned}$$

The first equality comes directly from the Itō isometry while the second is just a change of variable. It follows that for any $T > 0$, there exist a constant C such that the above expectation is bounded above by C uniformly as $t \in [0,T]$. This in turn yields that, for $p > 0$, there is a constant C_p such that $\mathbb{E}\left[\int_0^T \|y(t)\|^p \, dt\right] < C_p$ by Fernique's theorem, i.e. $y(t) \in L^p([0,T], \mathcal{H})$ almost surely.

Recall that a mild solution is, by definition, of the form $u(t) = S(t)u_0 + \int_0^t S(t-s)Q \, dW_s$. Using the integral identity $\int_s^t (t-r)^{a-1}(r-s)^{-\alpha} \, dr = \frac{1}{c_\alpha}$ for $s < t$, $\alpha > 0$ and c_α a constant, we get that

$$\begin{aligned}u(t) &= S(t)u_0 + c_\alpha \int_0^t \int_s^t (t-r)^{\alpha-1}(r-s)^{-\alpha} S(t-s) \, dr \, Q \, dW_s \\ &= S(t)u_0 + c_\alpha \int_0^t \int_0^r (t-r)^{\alpha-1}(r-s)^{-\alpha} S(t-s)Q \, dW_s \, dr \\ &= S(t)u_0 + c_\alpha \int_0^t (t-r)^{\alpha-1} S(t-r) \int_0^r (r-s)^{-\alpha} S(r-s)Q \, dW_s \, dr. \\ &= S(t)u_0 + c_\alpha \int_0^t (t-r)^{\alpha-1} S(t-r) y(r) \, dr\end{aligned}$$

Thus, what we said above is that we want to show that

$$y \mapsto S(t)u_0 + c_\alpha \int_0^t (t-r)^{\alpha-1} S(t-r) y(r) \, dr$$

goes from $\mathbb{L}^p([0,T], \mathcal{H})$ to $\mathcal{C}([0,T], \mathcal{H})$. Fernique's theorem gave us that $y(t) \in L^p([0,T], \mathcal{H})$. Now, we assumed that the semigroup $S(t)$ is strongly continuous so we only have to check whether $\int_0^t (t-r)^{\alpha-1} S(t-r) y(r) \, dr$ is itself continuous.

Note that $t \mapsto S(t)$ is uniformly bounded and that if $\alpha \in (0, \frac{1}{2})$, then $t \mapsto (t-r)^{\alpha-1} \in L^q$ for $q \in [1, \frac{1}{1-\alpha})$. This allows us to use Hölder inequality to get

$$\sup_{t\in[0,T]} \| \int_0^t (t-r)^{\alpha-1} S(t-r) y(r) \, dr \|^p \le C_T \int_0^T \|y(t)\|^p \, dt$$

when $p > \frac{1}{\alpha}$. By the density of continuous functions in L^p, this gives us that $\int_0^t (t-r)^{\alpha-1} S(t-r) y(r) \, dr$ is continuous for any function $y(t) \in L^p$ if it is continuous for any $y(t)$ that is itself continuous.

Let $F_y(t) = \int_0^t (t-r)^{\alpha-1} S(t-r) y(r) \, dr$. Then applying HÖlder inequality to $\|F_y(t+h) - F_y(t)\|$, then the dominated convergence theorem and finally taking the limit as $h \to 0$ gives that $F_y(t)$ is right continuous when $y(t)$ is itself continuous. Similarly we get that $F_y(t)$ is left continuous and we are done. $\square$

To prove regularity in space, we use the norm $\|\cdot\|_{HS}$. This norm is the Hilbert-Schmidt norm and is defined by $\|A\|_{HS}^2 = \operatorname{tr} AA^*$ for A a linear operator. We also use $\mathcal{H}_\alpha$ to denote the interpolation spaces corresponding to the generator L.

THEOREM 9 (Spatial regularity). *If there exist $\alpha \geq 0$ and $\beta \in (0, \frac{1}{2} + \alpha)$ such that $Q : \mathcal{H} \to \mathcal{H}_\alpha$ is bounded and $\|(-L)^{-\beta}\|_{HS} < \infty$ for L the generator of an analytic semigroup, then the solution u to the associated stochastic linear equation (†) takes values in $\mathcal{H}_\gamma$ for every $\gamma < \frac{1}{2} + \alpha - \beta$.*

PROOF. We want to show that

$$\int_0^T \|(-L)^\gamma S(t) Q\|_{HS}^2 \, dt < \infty \text{ for all } T > 0.$$

Because we are assuming that $Q : \mathcal{H} \to \mathcal{H}_\alpha$ is bounded, we have that there exists a constant $C > 0$ such that $\|Q\|_{HS}^2 \leq C\|(-L)^{-\alpha}\|_{HS}^2$. This yields that

$$\begin{aligned}
\int_0^T \|(-L)^\gamma S(t) Q\|_{HS}^2 \, dt &\leq C \int_0^T \|(-L)^\gamma S(t)(-L)^{-\alpha}\|_{HS}^2 \, dt \\
&= C \int_0^T \|(-L)^{\gamma-\alpha} S(t)\|_{HS}^2 \, dt.
\end{aligned}$$

Because $\|(-L)^{-\beta}\|_{HS} < \infty$, we have that $(-L)^{-\beta}$ is an Hilbert-Schmidt operator and thus

$$\begin{aligned}
\|(-L)^{\gamma-\alpha} S(t)\|_{HS} &= \|(-L)^{-\beta}(-L)^{\beta+\gamma-\alpha} S(t)\|_{HS} \\
&\leq \|(-L)^{-\beta}\|_{HS} \|(-L)^{\beta+\gamma-\alpha} S(t)\|.
\end{aligned}$$

Using the theorem that we have in section 3 on the smoothing effect of a semigroup, we have that $\|(-L)^{\beta+\gamma-\alpha} S(t)\| \leq C t^{\alpha-\beta-\gamma}$ for some constant C. Putting everything together and using that $\|(-L)^{-\beta}\|_{HS} < \infty$ by assumption, we get that

$$\|(-L)^{\gamma-\alpha} S(t)\|_{HS} \leq C \max(1, t^{\alpha-\beta-\gamma})$$

for some constant C. It will follow that $\int_0^T \|(-L)^\gamma S(t) Q\|_{HS}^2 \, dt < \infty$ for $T > 0$ when $t^{\alpha-\beta-\gamma}$ is integrable near $t = 0$, i.e. when $\alpha - \beta - \gamma > -\frac{1}{2}$. We conclude that $u \in \mathcal{H}_\gamma$ for $\gamma < \frac{1}{2} + \alpha - \beta$. $\square$

THEOREM 10 (Regularity in time). *Under the same assumptions as in the last theorem on spatial regularity, fix $\gamma < \frac{1}{2} + \alpha - \beta$. For every $t > 0$, the process u is almost surely Hölder-δ continuous in $\mathcal{H}_\gamma$ for every $\delta <$* $\min(\frac{1}{2}, \frac{1}{2} + \alpha - \beta - \gamma)$, PROOF. The proof is an application of Kolmogorov's continuity theorem. We want to show that

$$\mathbb{E}\left[\|u(t)-u(s)\|_\gamma^2\right] \le C|t-s|^{\min(1,2(\bar{\gamma}-\gamma))}$$

uniformly for s,t in $[t_0,T]$, for every $0<t_0<T$ and every $\bar{\gamma}<\frac{1}{2}+\alpha-\beta$. To this end, we recover the Markov property from the mild solution by using the semigroup property and the independence of the increments of the stochastic integral of the mild solution u. That is, we have

$$u(t) = S(t-s)u(s) + \int_s^t S(t-r)Q\,dW_r$$

almost surely when $s<t$. Now, a basic result in probability gives that $u(s)$ is independent of the increments of W on $[s,t]$ which gives us the orthogonal decomposition

$$\mathbb{E}\left[\|u(t)-u(s)\|_\gamma^2\right] = \mathbb{E}\left[\|S(t-s)u(s)-u(s)\|_\gamma^2\right] + \int_0^{t-s} \|(-L)^\gamma S(r)Q\|_{HS}^2\,dr.$$

Note that the Itō isometry is again used on the second term of the above equality. From section 3, we have a theorem on the decay of an analytic semigroup. This theorem gives us that

$$\mathbb{E}\left[\|S(t-s)u(s)-u(s)\|_\gamma^2\right] \le C|t-s|^{\min(1,2(\bar{\gamma}-\gamma))}\mathbb{E}\left[\|u(s)\|_\gamma^2\right]$$

for some constant $C>0$. The same argument as in the previous theorem gives that

$$\int_0^{t-s} \|(-L)^\gamma S(r)Q\|_{HS}^2\,dr \le C\int_0^{t-s} \max(1, r^{\alpha-\beta-\gamma})^2\,dr.$$

Adding the two terms together yields

$$\mathbb{E}\left[\|u(t)-u(s)\|_\gamma^2\right] \le C|t-s|^{\min(1,2(\bar{\gamma}-\gamma))}$$

and thus we can apply Kolmogorov's continuity theorem as desired. □

4.3. Globality of the solutions.

Our last building bloc from the theory of linear stochastic partial differential equations is what we mean by a long term behavior to such equations. In the stochastic case, a long term behavior is model by an invariant measure on the law of the process $S(t)u + \int_0^t S(t-s)Q\,dW_s$.

The theory goes far beyond the level of this class so let just state one important theorem that guarantee the existence of such an invariant measure.

THEOREM 11 (Existence of an invariant measure). *Define the self-adjoint operator $Q_t : \mathcal{H}\to\mathcal{H}$, for $\mathcal{H}$ an Hilbert space where (†) lives, by*

$$Q_t = \int_0^t S(t)QQ^*S^*(t)dt.$$

Then there exist an invariant measure for (†) *if and only if*

$$\sup_{t>0} \operatorname{tr} Q_t < \infty.$$

5. Semilinear SPDEs

We have now reach the general theory that will enable us to solve the Navier-Stokes equations with a stochastic force on the two-dimensional torus $\mathbb{T}^2$.

Similarly to the linear case, we start with the following two definitions.

DEFINITION 8 (Stochastic semilinear equation). *By a stochastic semilinear equation on a separable Banach space $\mathcal{B}$, we mean an equation of the form*

$$\begin{cases} du = Lu\,dt + F(u)\,dt + Q\,dW_t, \\ u(0) = u_0 \in \mathcal{B} \end{cases} \tag{‡}$$

where L is the generator of a strongly continuous semigroup on $\mathcal{B}$, $Q : \mathcal{H} \to \mathcal{B}$ is a bounded linear operator and W is a cylindrical Wiener process on the Hilbert space $\mathcal{H}$. We also assume that F is measurable from the linear subspace $\mathcal{D}(F) \subset \mathcal{B}$ into $\mathcal{B}$.

DEFINITION 9 (Mild solution). *We say that a $\mathcal{B}$-valued process $t \mapsto u(t) \in \mathcal{D}(F)$ is a mild solution to* (‡) *if*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\,ds + \int_0^t S(t-s)Q\,dW_s$$

almost surely for every $t > 0$.

When the equation is stochastic and semilinear, we can't find much about the maximal time of existence to the solution. It may be infinite if the solution is global and it could be finite if there is a finite time blow up. But this is not all! There is an interesting third possibility that the stochastic linear equations or the deterministic semilinear equations don't have. That is, the maximal time of existence could depend on the sample space and thus be completely random. If this happens, the same solution to (‡) could sometimes be global in time and other times have a finite time blow up. Moreover, there could be an arbitrary number of different finite times of existence.
To help deal with all those possibilities, we introduce the new notion of a local mild solution. For a recall on what a stopping time is, please see the appendix.

DEFINITION 10 (Local mild solution). *A $\mathcal{D}(F) \subset \mathcal{B}$-valued stochastic process $t \mapsto u(t)$ together with a stopping time τ satisfying the following two conditions*

(A) $\tau > 0$ *almost surely,*

(B) $$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\,ds + \int_0^t S(t-s)Q\,dW_s$$

is called a local mild solution to (‡) if condition (B) holds almost surely for every time t such that $t < \tau$ almost surely. In this case, we simply say that (u, τ) is a local mild solution to (‡)

Adding the stopping time allows us to get rid of the multiple cases where the solution may be sometimes define and sometimes not. Stopping time are extremely useful in probability to get rid of multiple random cases and thus getting an object that will necessarily satisfies an interesting property that we are interested in.

DEFINITION 11 (Local maximal mild solution). *We say that a local mild solution (u, τ) is maximal if, given any local mild solution $(\tilde{u}, \tilde{\tau})$, we have $\tilde{\tau} \leq \tau$ almost surely.*

The first theorem of this section will allows us to solve (‡) for a unique local maximal mild solution when the nonlinearity $F : \mathcal{B} \to \mathcal{B}$ is assume to be Lipschitz continuous. Furthermore, this solution will have a some regularity properties.

THEOREM 12 (Existence and uniqueness, Lipschitz case). *If the nonlinearity $F : \mathcal{B} \to \mathcal{B}$ is Lipschitz continuous when restricted to any bounded set, i.e. locally Lipschitz continuous, then there exists a unique local maximal mild solution to (‡) if we assume that the convolution $\int_0^t S(t-s)Q\,dW_s$ has, almost surely, continuous $\mathcal{B}$-valued sample paths.*

PROOF. For $T > 0$ and a function $g \in \mathcal{C}(\mathbb{R}_+, \mathcal{B})$, we define a map $M_{g,T} : \mathcal{C}([0,T], \mathcal{B} \to) \mathcal{C}([0,T], \mathcal{B}$ by

$$(M_{g,T}u)(t) = \int_0^t S(t-s)F(u(s))\,ds + g(t).$$

By increasing F if need be, we can assume that $\|S(t)\| \leq M$ for some constant M, therefore we get

$$\sup_{t\in[0,T]} \|M_{g,T}u(t) - M_{g,T}v(t)\| \leq MT \sup_{t\in[0,T]} \|F(u(t)) - F(v(t))\|.$$

And similarly

$$\sup_{t\in[0,T]} \|M_{g,T}u(t) - g(t)\| \leq MT \sup_{t\in[0,T]} \|F(u(t))\|.$$

Because F is locally Lipschitz, for any constant $R > 0$, the above two inequalities yield that there is a maximal $T > 0$ such that $M_{g,T}$ is a contraction from the ball of radius R centered ar g in $\mathcal{C}([0,t],\mathcal{B})$ to itself. By the Banach fixed point theorem, $M_{g,T}$ must have a unique fixed point when T is bounded above by the maximal T of the preceding sentence.

Let $W_L \in \mathcal{C}(\mathbb{R}_+,\mathcal{B})$ be one realisation of the stochastic convolution $\int_0^t S(t-s)F(u(s))\,ds$. Because we assume that the path of the convolution are continuous we can let $g(t) = S(t)u_0 + W_L(t)$. We get that the pair (u,T) is a single path satisfying part (B) of the definition of a local mild solution.

Solving the equation for every realisation of the stochastic convolution and letting τ equals to maximal time of existence of each realisation, we get that the pair (u,τ) is a local mild solution to (†).

As in the deterministic case, we can check that the local mild solution thus constructed can be extended to a local maximal mild solution. The uniquness follows also from the pathwise uniqueness which we know we have from the uniqueness in the deterministic case. □

Corollary 1 (Blow up criterion). *The above local maximal mild solution to (‡) has almost surely continuous sample paths. Furthermore,*

$$\lim_{t\uparrow\tau} \|u(t)\| = \infty, \quad \textit{almost surely,}$$

on the set $\{\tau < \infty\}$. If $F : \mathcal{B} \to \mathcal{B}$ is globally Lipschitz continuous, then $\tau = \infty$ almost surely. In other words, the solution is global in time.

Proof. We know that the result is true pathwise from the deterministic theory. Because local maximal mild solution is a pathwise property, we get the result in the stochastic case. □

Unfortunately, the assumption that the nonlinearity is Lipschitz continuous is too strong for us. We need a more powerful result. The following theorem closely match the previous, however we assume that F is less regular than Lipschitz continuous in $\mathcal{B}$ (locally Lipschitz continuous in an interpolation space $\mathcal{B}_\beta$). To still get similar conclusion, we need regularity elsewhere. We will thus assume that the infinitesimal generator L generated a semigroup that is not only strongly continuous, but also analytic.

Theorem 13 (Existence and uniqueness, analytic semigroup). *Let L be the infinitesimal generator of an analytic semigroup on the Banach space $\mathcal{B}$. Let the stochastic convolution $\int_0^t S(t-s)Q\,dW_s$ have, almost surely, continuous sample paths in one interpolation space $\mathcal{B}_\alpha$ for $\alpha \geq 0$.*

If there exists $\gamma \geq 0$ and $\delta \in [0,1)$ such that, for every $\beta \in [0,\gamma]$, the nonlinearity $F : \mathcal{B} \to \mathcal{B}$ extends to a locally Lipschitz continuous map $F : \mathcal{B}_\beta \to \mathcal{B}_{\beta-\delta}$ of at most polynomial order growth, then there exists a unique local maximal mild solution (u,τ) to (‡). Furthermore, $u \in \mathcal{B}_\beta$ for every $\beta < \min(\alpha, 1+\gamma-\delta)$.

PROOF. For $\delta > 0$, we can bound $\|S(t-s)F(u(s))\|$ in term of $\|F(u(s))\|_{-\delta}$ to get

$$\sup_{t\in[0,T]} \|M_{g,T}u(t) - M_{g,T}v(t)\| \leq MT^{1-\delta} \sup_{t\in[0,T]} \|F(u(t)) - F(v(t))\|$$

and

$$\sup_{t\in[0,T]} \|M_{g,T}u(t) - g(t)\| \leq MT^{1-\delta} \sup_{t\in[0,T]} \|F(u(t))\|$$

where $M_{g,T}$ is defined as in the locally Lipschitz case. This immeditatly implies the existence of a unique local maximal mild solution $\mathcal{B}$-valued process (u,τ) using the Banach-Fixed point theorem and the usual argument.

The hard part of the theorem is about the reguliarity statement $u \in \mathcal{B}_\beta$ for $t < \tau$ and $\beta \leq \min(\alpha,\gamma)$. To bound $\|u_t\|_\beta$, we will show that for every $\beta \leq \min(\alpha,\gamma)$, there exist exponenent $p_\beta \geq 1$, $q_\beta \geq 0$ and constants $a \in (0,1)$ and $C > 0$ such that

$$\|u_t\|_\beta \leq Ct^{-q_\beta} \left(1 + \sup_{s\in[at,t]} \|u_s\| + \sup_{s\in[0,t]} \|W_L^a(s)\|_\beta\right)^{p_\beta}$$

where $W_L^a(t) = \int_{at}^t S(t-r)Q\,dW_r$ for $a \in [0,1)$ a parameter. Note that $W_L^a(t) = W_L(t) - S((1-a)t)W_L(at)$ so if we assume that $W_L(t) = \int_0^t S(t-s)Q\,dW_s$ has almost surely continuous sample paths and that it lives in some interpolation space $\mathcal{B}_\alpha$, the same is also true for $W_L^a(t)$.

Note that the inequality is trivially true for $\beta = 0$, $p_\beta = 1$ and $q_\beta = 0$. We will show that if the inequality is true for some $\beta_0 \in [0,\gamma]$ then, for any $\epsilon \in (0, 1-\delta)$, it will also be true for $\beta = \beta_0 + \epsilon$ if we change the constants in the expression. Because we can go from $\beta = 0$ to $\beta < \gamma + 1 - \delta$ using a finite number of such step, it will prove the inequality for all β and thus end the proof.

Using the definition of a local mild solution together with the particular form of $W_L^a(t)$, one find that

$$u_t = S((1-a)t)u_{at} + \int_{at}^t S(t-s)F(u(s))\,ds + W_L^a(t).$$

Now, since $\beta \leq \gamma$ and since F has polynomial growth, there exists an integer $n > 0$ such that

$$\begin{aligned}\|u_t\|_{\beta+\epsilon} &\leq Ct^{-\epsilon}\|u_{at}\|_\beta + \|W_L^a(t)\|_{\beta+\epsilon} + C\int_{at}^t (t-s)^{-(\epsilon+\delta)}(1+\|u_s\|_\beta)^n\,ds \\ &\leq C(t^{-\epsilon} + t^{1-\epsilon-\delta}) \sup_{s\in[at,t]} (a + \|u_s\|_\beta^n) + \|W_l^a(t)\|_{\beta+\epsilon} \\ &\leq Ct^{-\epsilon} \sup_{s\in[at,t]} (a + \|u_s\|_\beta^n) + \|W_L^a(t)\|_{\beta+\epsilon}.\end{aligned}$$

Using the induction hypothesis, i.e. that the inequality is true for β, we get for $\beta + \epsilon$

$$\|u_t\|_{\beta+\epsilon} \leq Ct^{-\epsilon - nq_\beta}(1 + \sup_{s\in[a^2t,t]} \|u_s\| + \sup_{s\in[a^2t,t]} \|W_L^a(s)\|_\beta)^{np_\beta} + \|W_L^a(t)\|_{\beta+\epsilon}.$$

It follows that we can change the constants so that the inequality is true for any $\beta < \gamma + 1 - \delta$. $\square$

6. Stochastic Navier-Stokes Equations

As said in the introduction, we want to know more about solutions to the system

$$\begin{cases} du = \Delta u\, dt - (u\cdot\nabla)u\, dt - \nabla p\, dt + Q\, dW_t, \\ \operatorname{div} u = 0 \end{cases}$$

Because this system is not directly of the semilinear form that we have studied, we want to rewrite it before any attempt to solve it. We thus introduce the Leray projector which is the same operator that we have seen in class.

Definition 12 (Leray projector). *We call the operator Π acting as the orthogonal projection onto the space of divergence-free vector fields the Leray projector. In Fourier space, this operator is given by*

$$(\Pi u)_k = u_k - \frac{k\langle k, u_k\rangle}{|k|^2}.$$

Under this transformation, the stochastic Navier-Stokes equations become the following semilinear equation:

$$\text{(N-S)} \qquad \begin{cases} du = \Delta u\, dt - \Pi(u\cdot\nabla)u\, dt + Q\, dW_t, \\ \operatorname{div} u = 0 \end{cases}$$

From now on, we will write $F(u) = \Pi(u\cdot\nabla)u$ to denote the nonlinearity.

Theorem 14 (Existence and uniqueness of a local maximal mild solution for the stochastic Navier-Stokes equations).

Proof. Note first that the Leray projector is a contraction in the Sobolev space $H^s(\Pi^2, \mathbb{R}^2)$ for any s. Using the Banach algebra property of Sobolev space for $s \geq 0$, we thus get that

$$\|F(u)\|_{H^s} \leq \|u\|_{H^r}\|\nabla u\|_{H^q} \leq C\|u\|^2_{H^r}.$$

when $r > \max(s, \frac{s}{2}+1)$. If $s > 0$, we can use $r = s+1$. Because the infinitesimal generator L is equal to the Laplacian Δ which is a polynomial differential operator of degree 2, we can use our lemma for such operator (from section 2) on the Hilbert space $\mathcal{H} = H^s$ to get that the interpolation spaces are given by $\mathcal{H}_\alpha = H^{s+2\alpha}$.

Now, provided that $s > 1$, we can use Theorem 12 on the existence and uniqueness of local maximal mild solution to semilinear SPDEs with $\gamma = s$. By this theorem, we will even get that $u \in H^{s+2\beta}$ if $\beta < \min(\alpha, s)$ where α is the maximal order for which the stochastic convolution $\int_0^t S(t-s)Q\,dW_s$ is in H^α when the semigroup $S(t)$ is generated by the Laplacian). This yields that the maximal solution u is in Sobolev spaces of order as high as the Wiener measure allows. That is, the solution is as smooth as could be possible. □

While the existence and uniqueness of a local maximal mild solution follows easily from our main theorem for semilinear SPDEs, this theorem does not tell us anything about the globality of such a maximum solution. This globality property is the hardest question surrounding the Navier-Stokes equations. To prove that the local maximal mild solution is in fact global, we will find an *a priori* estimate on the solution by introducing a vorticity to the system.

THEOREM 15 (Global maximal solution). *The solution to* (N-S) *is global in time.*

PROOF. We try to get bounds on $F(u)$ in negative order Sobolev spaces using the fact that H^{-s} can be identified with the dual of H^s so that

$$\|F(u)\|_{H^{-s}} = \sup\left\{\int F(u)(x)v(x)\,dx, \quad v \in \mathcal{C}^\infty, \quad \|v\|_{H^s} \leq 1\right\}.$$

Using that the solution is divergence free, we get that

$$\int F(u)(x)v(x)dx = -\int v_j u_i \partial_i u_j\,dx \leq \|v\|_{L^p}\|\nabla u\|_{L^2}\|u\|_{L^q}$$

for $p, q > 2$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. From $\|u\|_{L^q} \leq C_q\|\nabla u\|_{L^2}$ for every $q \in [2, \infty)$, we can conclude that for every $s > 0$, there is a constant $C > 0$ such that

$$\|F(u)\|_{H^{-s}} \leq C\|\nabla u\|_{L^2}^2.$$

To get *a priori* bounds, we introduce the vorticity $w = \nabla \wedge u = \partial_1 u_2 - \partial_2 u_1$. Note that we can assume that the range of Q consists of mean zero vector fields by translation, and thus we can assume that the solution u to (N-S) has mean zero. In this mean zero situation, the vorticity w is enough to describe u completely because $\operatorname{div} u = 0$. To see this, remark that we have the map $w \mapsto u$ given by

$$u_k = (Kw)_k = \frac{k^\perp w_k}{|k|^2}, \qquad (k_1, k_2)^\perp = (-k_2, k_1).$$

This explicit formula shows that K is a bounded operator from H^s to H^{s+1} for any $s > 0$. Thus, we can rewrite the stochastic Navier-Stokes equations to get that

$$dw = \Delta w\,dt + \tilde{F}(w)\,dt + \tilde{Q}\,dW_t \qquad \text{(N-S, } w\text{)}$$

for $\tilde{F}(w) = \nabla \wedge F(Kw)$. Using the H^{-s} bound for $F(u)$, we get that

$$\|\tilde{F}(w)\|_{H^{-1-s}} \leq C\|w\|_{L^2}^2.$$

This means that F is locally Lipschitz from L^2 to H^s for every $s < -1$. From the existence and uniqueness theorem for locally Lipschitz function (theorem 11), we get that (N-S, w) admits a unique solution for any initial condition in L^2.

Define $\tilde{W}_L(t)$ to be the stochastic convolution $\int_0^t e^{\Delta(t-s)}\tilde{Q}\, dW_s$ and let $v(t) = w(t) - \tilde{W}_L(t)$. Using this notation, we have that v is the unique solution to the random equation $\partial_v = \Delta v + \tilde{F}(v + \tilde{W}_L)$. We know that $\|\tilde{F}(w)\|_{H^{-s}} \leq C\|w\|_{H^s}^2$ for $s > \frac{1}{3}$. We can assume that $\tilde{W}_l$ is in $H^{\frac{1}{2}}$ almost surely. Now, we use the divergence free assumption to get the L^2 bound

$$\begin{aligned}
\partial_t\|v\|^2 &= -2\|\nabla v\|^2 - 2\langle \tilde{W}_L, \tilde{F}(v + \tilde{W}_L)\rangle \\
&\leq -2\|\nabla v\|^2 + 2\|\tilde{W}_L\|_{H^{\frac{1}{2}}}\|v + \tilde{W}_L\|_{H^{\frac{1}{2}}}^2 \\
&\leq -2\|\nabla v\|^2 + 4\|\tilde{W}_L\|_{H^{\frac{1}{2}}}\left(\|v\|_{H^{\frac{1}{2}}}^2 + \|\tilde{W}_L\|_{H^{\frac{1}{2}}}^2\right) \\
&\leq -2\|\nabla v\|^2 + 4\|\tilde{W}_L\|_{H^{\frac{1}{2}}}\left(\|v\|\|\nabla v\| + \|\tilde{W}_L\|_{H^{\frac{1}{2}}}^2\right) \\
&\leq 8\|\tilde{W}_L\|_{H^{\frac{1}{2}}}^2\|v\|^2 + 2\|\tilde{W}_L\|_{H^{\frac{1}{2}}}^3.
\end{aligned}$$

Gronwall's inequality now gives that the solution is global in time. □

Appendix A. Probability

Definition 13 (Stopping time). *A stopping time $\tau : \Omega \to \mathbb{R}_+$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+,}, \mathbb{P})$ is a random variable such that the event $\{\tau \leq t\}$ is adapted to the filtration $\mathcal{F}_t$ for any $t \in \mathbb{R}_+$.*

Definition 14 (Markov's property). *A stochastic process $X : \mathbb{R}_+ \times \Omega \to \mathcal{B}$ is said to satisfies the Markov's property if*

$$\mathbb{E}^{X_0}\left[f(X_{t+h})|\mathcal{F}_t\right] = \mathbb{E}^{X_t}\left[f(X_h)\right]$$

for any measurable function f.

Theorem 16 (Kolmogorov's continuity theorem). *Let $X : \mathbb{R}_+ \times \Omega \to \mathcal{B}$ be a stochastic process such that, for every $T > 0$, there exist constants $\alpha, \beta, C > 0$ with*

$$\mathbb{E}\left[\|X_t - X_s\|^\alpha\right] \leq C|t-s|^{1+\beta}$$

for $s, t \in [0, T]$. Then X has a modification with almost surely continuous sample paths.

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