MATH 581 ASSIGNMENT 4

DUE FRIDAY MARCH 13

1. For $u \in \mathscr{D}'$ and $v \in \mathscr{E}'$, prove the following.
   (a) $\operatorname{sing\,supp}(u * v) \subset \operatorname{sing\,supp}(u) + \operatorname{sing\,supp}(v)$.
   (b) $\mathrm{WF}(u * v) = \{(x + y, \xi) : (x, \xi) \in \mathrm{WF}(u) \text{ and } (y, \xi) \in \mathrm{WF}(v)\}$.
2. Compute, as explicitly as you can, the fundamental solutions $E^{\pm}$ of the wave operator $\Box = \partial_n^2 - \partial_1^2 - \ldots - \partial_{n-1}^2$, satisfying $\operatorname{supp} E^{\pm} \subset \overline{\mathbb{R}^n_{\pm}}$. Note that these fundamental solutions are unique, since $\operatorname{supp} \delta \subset \overline{\mathbb{R}^n_+}$ and $\operatorname{supp} \delta \subset \overline{\mathbb{R}^n_-}$. Determine $\mathrm{WF}(E^{\pm})$.
3. Let $u \in \mathscr{D}'$ be a solution of $\Box u = 0$. Show that if
$$Q_0 = (x_0, t_0, \xi_0, \tau_0) \in \mathrm{WF}(u),$$
then $\tau_0 = \pm|\xi_0|$ and
$$Q_s = (x_0 \pm \tfrac{\xi_0}{|\xi_0|} s, t_0 + s, \xi_0, \tau_0) \in \mathrm{WF}(u),$$
for all small values of $s \in \mathbb{R}$. Note that $Q_0 \mapsto Q_s$ is the Hamiltonian flow (i.e., bicharacteristic strip) corresponding to the symbol of $\Box$. *Hint*: Consider $\phi \in \mathscr{D}$ with $\phi \equiv 1$ near $(x_0, t_0) \in \mathbb{R}^n$, and invoke $\phi u = E^{\pm} * (\Box(\phi u))$.
4. Let $P(D) = [p_{jk}(D)]$ be a square matrix consisting of constant coefficient linear partial differential operators. Show that $P(D)$ admits a fundamental matrix supported in a cone C satisfying $C \cap \overline{\mathbb{R}^n_-} = \{0\}$ if and only if the scalar operator $\det P(D)$ is hyperbolic in the sense of Gårding. *Hint*: The cofactor matrix.
5. For $s \in \mathbb{R}$, the (Bessel potential) *Sobolev space* $H^s(\mathbb{R}^n)$ is the set of those $u \in \mathscr{S}'(\mathbb{R}^n)$ with $\|u\|_{H^s} := \|\langle D \rangle^s u\|_{L^2} < \infty$, where the *Bessel potential* $\langle D \rangle^s u$ of u is defined by
$$\widehat{\langle D \rangle^s u}(\xi) = \langle \xi \rangle^s \hat{u}(\xi) \equiv (1 + |\xi|^2)^{s/2} \hat{u}(\xi).$$
Prove the following.
   (a) $\langle D \rangle^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert space isometry.
   (b) For $k \geq 0$ integer, $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.
   (c) $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.
   (d) The (topological) dual of $H^s(\mathbb{R}^n)$ is isometric to $H^{-s}(\mathbb{R}^n)$.
   (e) The *trace operator* $\gamma : \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^{n-1})$ defined by
$$(\gamma u)(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{n-1}, 0),$$
   has a unique extension to a bounded linear operator $\gamma : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.

Date: Winter 2020.