MATH 581 ASSIGNMENT 3

DUE FRIDAY FEBRUARY 28

1. Let $u \in \mathscr{E}'(\mathbb{R}^n)$ satisfy $p(\partial)u = 0$ in a bounded domain $\Omega \subset \mathbb{R}^n$, and let $E \in \mathscr{D}'(\mathbb{R}^n)$ be a fundamental solution of $p(\partial)$. Pick an arbitrary $y \in \Omega$, and consider the ball $B = B_r(y) \subset \Omega$ with $r > 0$ small. We want to express $u|_B$ in terms of the values of u near the boundary $\partial\Omega$, generalizing the Green formulas. Let $U \subset \mathbb{R}^n$ be an open set containing the origin, and let $\zeta \in \mathscr{D}(U + B_\varepsilon)$ be a cut-off function with $\zeta \equiv 1$ in U, with $\varepsilon > 0$ small. Prove the representation formula
$$u = K * u \qquad \text{in } B,$$
where $K = p(\partial)[(1-\zeta)E]$, and we have assumed that $B - (U + B_\varepsilon) \subset \Omega$. Prove also the following.
   (a) If $u = 0$ in a neighbourhood of $\partial\Omega$, then $u = 0$ in Ω. *Hint*: What is $\operatorname{supp} K$?
   (b) If u is smooth in a neighbourhood of $\partial\Omega$, then u is smooth in Ω.
2. Does there exist a Gevrey hypoelliptic operator of exponent m, for every natural number m? Give an explicit example when there is one.
3. Recall that by Hörmander's theorem, $p(D)$ is hypoelliptic if and only if for any $\eta \in \mathbb{R}^n$ one has $p(\xi + i\eta) \neq 0$ for all sufficiently large $\xi \in \mathbb{R}$.
   (a) Construct a non-hypoelliptic polynomial p in dimension $n > 1$ such that $|p(\xi)| \to \infty$ as $|\xi| \to \infty$ for $\xi \in \mathbb{R}^n$.
   (b) For any given $c > 0$, construct a non-hypoelliptic polynomial p in dimension $n > 1$ such that $|p(\xi + i\eta)| \to \infty$ uniformly in $\{|\eta| \leq c\}$ as $|\xi| \to \infty$ for $\xi \in \mathbb{R}^n$.
4. In this exercise we will refine the hypoellipticity and microlocal regularity theorems we have seen in class. First we localize the notion of Sobolev regularity in space and frequency. With $\Omega \subset \mathbb{R}^n$ a domain, let $u \in \mathscr{D}'(\Omega)$, and let $(x_0, \xi_0) \in \Omega \times \mathbb{R}^\times_n$, where $\mathbb{R}^\times_n = \mathbb{R}^n \setminus \{0\}$. We write $u \in H^s(x_0)$ if there is $\phi \in \mathscr{D}(\Omega)$ with $\phi(x_0) \neq 0$ such that $\phi u \in H^s(\mathbb{R}^n)$. Similarly, we write $u \in H^s(x_0, \xi_0)$ if there is $\phi \in \mathscr{D}(\Omega)$ with $\phi(x_0) \neq 0$, and there is a conical neighbourhood V of ξ_0 such that $(1 + |\cdot|)^s \widehat{\phi u}$ is square integrable in V. With these localizations at hand, we define the singular support
$$\operatorname{sing\,supp}^s(u) = \Omega \setminus \{x \in \Omega : u \in H^s(x)\},$$
and the wave front set
$$\mathrm{WF}^s(u) = \Omega \times \mathbb{R}^\times_n \setminus \{(x, \xi) : u \in H^s(x, \xi)\},$$
adapted to Sobolev regularity. Prove the following.

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(a) Let p be a polynomial of degree m in $\mathbb{R}^n$, satisfying

$$|\xi|^\gamma \lesssim \mu_p(\xi), \qquad \xi \in \mathbb{R}^n,$$

for some constant $0 < \gamma \leq 1$, where

$$\mu_p(\xi) = \inf\{|\eta| : p(\xi + i\eta) = 0, \, \eta \in \mathbb{R}^n\}.$$

Then

$$\operatorname{sing\,supp}^{s+\gamma m}(u) \subset \operatorname{sing\,supp}^s(p(D)u), \qquad u \in \mathscr{D}'(\Omega).$$

(b) Let P be a differential operator of order m with smooth coefficients in Ω. Then

$$\mathrm{WF}^{s+m}(u) \subset \mathrm{Char} P \bigcup \mathrm{WF}^s(Pu), \qquad u \in \mathscr{D}'(\Omega).$$