MATH 581 ASSIGNMENT 2

DUE FRIDAY FEBRUARY 14

1. (a) Let $u \in \mathscr{E}'$ and $v \in \mathscr{D}'$. Show that
$$\tau_a(u * v) = (\tau_a u) * v = u * (\tau_a v), \qquad a \in \mathbb{R}^n,$$
where for distributions, the translation is defined by
$$\langle \tau_a u, \varphi \rangle = \langle u, \tau_{-a} \varphi \rangle.$$
   (b) For any distribution u, show that
$$\tau_a u = \delta_a * u,$$
where $\delta_a = \tau_a \delta$ is the Dirac mass concentrated at $a \in \mathbb{R}^n$.
2. (a) Show that
$$1 * (\delta' * \vartheta) \neq (1 * \delta') * \vartheta,$$
where 1 is the function identically 1 in $\mathbb{R}$, and ϑ is the Heaviside step function.
   (b) Let u and v be the surface measures of the spheres $\{x \in \mathbb{R}^3 : |x| = a\}$ and $\{|x| = b\}$, respectively. Compute $u * v$, and determine its singular support.
3. Prove that $\mathscr{D}(\Omega)$ is sequentially dense in $\mathscr{D}'(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open.
4. Construct a fundamental matrix for the *Stokes operator*
$$S : \begin{bmatrix} u \\ p \end{bmatrix} \mapsto \begin{bmatrix} -\Delta u + \nabla p \\ \nabla \cdot u \end{bmatrix},$$
where u is a vector field, and p is a scalar field in $\mathbb{R}^n$.
5. Prove a version of the Schwartz theorem for systems of equations. In particular, you need to extend the notions of hypoellipticity and analytic hypoellipticity to systems. Determine if the following systems are hypoelliptic/analytic hypoelliptic:
   - The Stokes system.
   - The div-curl system.
6. Let p be a nonzero polynomial. Show the following.
   (a) The equation $p(\partial)u = f$ has at least one smooth solution for every $f \in \mathscr{D}$.
   (b) If all solutions of $p(\partial)u = 0$ are smooth, then $p(\partial)$ is hypoelliptic.
   (c) If $p(\partial)$ admits a fundamental solution that is smooth outside some ball of finite radius (centred at the origin), then $p(\partial)$ is hypoelliptic.

Date: Winter 2020.