MATH 581 ASSIGNMENT 1

DUE FRIDAY JANUARY 31

1. Let $\varphi \in \mathscr{D}(\mathbb{R})$, $\varphi \neq 0$, and $\varphi(0) = 0$. In each of the following cases, decide if $\varphi_j \to 0$ as $j \to \infty$ in $\mathscr{D}(\mathbb{R})$. Does it hold $\varphi_j \to 0$ pointwise or uniformly?
   (a) $\varphi_j(x) = j^{-1}\varphi(x - j)$;
   (b) $\varphi_j(x) = j^{-n}\varphi(jx)$, where $n > 0$ is an integer.
2. In each case, show that f defines a distribution on $\mathbb{R}^2$, and find its order.
   (a) $f(\varphi) = \int_{\mathbb{R}^2} |x|^{-1} e^{|x|^2} \varphi(x) \mathrm{d}x$;
   (b) $f(\varphi) = \int_{\mathbb{R}} \varphi(s, 0) \mathrm{d}s$;
   (c) $f(\varphi) = \int_0^1 \partial_1 \varphi(0, s) \mathrm{d}s$.
3. Compute the following derivatives in the sense of distributions.
   (a) $\partial_x |x|$;
   (b) $\partial_x \log |x|$;
   (c) $\partial_2 f$, where $f \in \mathscr{D}'(\mathbb{R}^2)$ is the distribution from (b) of the previous exercise.
4. Prove the following.
   (a) $\partial_j(au) = (\partial_j a)u + a(\partial_j u)$ for $a \in C^\infty(\Omega)$ and $u \in \mathscr{D}'(\Omega)$.
   (b) There is no distribution on $\mathbb{R}$ such that its restriction to $\mathbb{R} \setminus \{0\}$ is $e^{1/x}$.
5. If u is the characteristic function of the unit ball in $\mathbb{R}^n$, compute $x \cdot \nabla u$.
6. Let $\omega \subset \mathbb{R}$ be an open interval, and $u \in \mathscr{D}'(\omega)$. Let $0 \leq k \leq \infty$.
   (a) Show that if $u' = 0$ then u is a constant function.
   (b) Show that if $u' = f$ with $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.
   (c) Show that if $u' + au = f$ with $a \in C^\infty(\omega)$ and $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.
7. Find the limits $n \to \infty$ of the following sequences in $\mathscr{D}'(\mathbb{R})$.
   a) $n\phi(nx)$, where ϕ is a nonnegative continuous function whose integral over $\mathbb{R}$ is finite.
   b) $n^k \sin nx$, where $k > 0$ is a constant.
   c) $x^{-1} \sin nx$.
8. For each of the following functions, determine if it is a tempered distribution, and if so compute its Fourier transform.
   (a) $x \sin x$,
   (b) $\frac{1}{x} \sin x$,
   (c) $e^{i|x|^2}$,
   (d) $x\vartheta(x)$, where ϑ is the Heaviside step function,

Date: Winter 2020.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.

Hand in your work in class, before the lecture on the due date. Email submissions are accepted only if you type your solutions in LaTeX. Please do not use any other means without discussing it with the instructor (In particular, we no not have a homework box). Please try not to use extensions but if you need, you can get individual extensions by sending me an email preferably well in advance of the due date stating your proposed new deadline.