BAIRE'S THEOREM AND ITS CONSEQUENCES

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Abstract. We prove Baire's theorem and its standard consequences: The uniform boundedness principle, the open mapping theorem, and the closed graph theorem.

Contents

1. Continuous maps

In a metric space X with metric ρ, we define the *ball* centered at $x \in X$, of radius r, to be the set $B_r(x) \equiv B(x,r) = \{y \in X : \rho(x,y) < r\}$. A subset $S \subseteq X$ is called *open* if for any $x \in S$, there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S$. Then S is closed iff its complement $X \setminus S$ is open. Indeed, S is *not closed* means that there is a sequence $\{x_n\} \subset S$ such that $x_n \to x \in X \setminus S$. On the other hand, $X \setminus S$ is *not open* means that there is some $x \in X \setminus S$ and a sequence $\{x_n\} \subset S$ such that $x_n \to x$. The following lemma clarifies to what extent continuity of a function is determined by the metrics we put on the domain and target spaces. In particular, it shows that continuity depends only on the piece of information contained in the metric that specifies which sets are open and which are not. Loosely speaking, continuity does not care about exact distances between points, it only cares about which points are infinitesimally close to each other (This of course leads to topology).

Lemma 1. *Let $\phi : X \to Y$ be a map between two metric spaces. Then the followings are equivalent.*

a) ϕ is continuous.
b) Whenever $U \subset Y$ is open, its preimage $\phi^{-1}(U) = \{x \in X : \phi(x) \in U\}$ is open.
c) The preimage of any closed $U \subset Y$ is closed.

Proof. The parts b) and c) are easily seen to be equivalent, since $\phi^{-1}(U) \cup \phi^{-1}(Y \setminus U) = X$ is a disjoint union. Now let ϕ be continuous, and let $U \subset X$ be closed. Suppose that $\{x_n\} \subset \phi^{-1}(U)$ is a sequence with $x_n \to x \in X$. Then from continuity we have $\phi(x_n) \to \phi(x)$, and from closedness of U we infer $\phi(x) \in U$. This establishes that a) implies c).

Suppose that b) holds. Then for any $\varepsilon > 0$ and $y = \phi(x)$ with $x \in X$, the preimage of $B_\varepsilon(y)$ contains a ball $B_\delta(x)$ with $\delta = \delta(\varepsilon, x) > 0$. In other words, the δ-closeness in X implies the ε-closeness in Y, which is continuity. □

It is immediate from the definition that the intersection of any collection of closed sets is again closed. The *closure* $\overline{S}$ of a subset $S \subset X$ is the intersection of all closed sets $C \subseteq X$ such that $C \supseteq S$.

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Exercise 1. Show that $\overline{S} = \{x \in X : \{x_n\} \subset S \text{ and } x_n \to x\}$.

2. The Baire category theorem

The following fundamental theorem is proved by René-Louis Baire in 1899.

Theorem 2 (Baire)**.** *Let X be a complete metric space, and let $\{C_n\}$ be a countable collection of closed subsets of X such that $\bigcup_n C_n = X$. Then at least one of C_n contains an open ball, i.e., there exist n, $x \in X$, and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset C_n$.*

Proof. Suppose that C_n does not contain any open ball, for any n. This means that any open ball B in X contains a point from $X \setminus C_n$, and so $B \cap (X \setminus C_n)$ contains a nontrivial closed ball, because $X \setminus C_n$ is open. Applying this with B equal to a ball of radius 1, we obtain $x_1 \in X$ and $r_1 \in (0, 1)$ such that $\overline{B(x_1, r_1)} \subset X \setminus C_1$. Similarly, there are $x_2 \in X$ and $r_2 \in (0, \frac{1}{2})$ such that $\overline{B(x_2, r_2)} \subset B(x_1, r_1) \cap (X \setminus C_2)$, and so on, we get a sequence of balls $B(x_n, r_n)$ such that $r_n \in (0, \frac{1}{n})$ and $\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1}) \cap (X \setminus C_n)$. In particular, we have $x_n \in \overline{B(x_k, r_k)}$ for $n > k$, hence $\{x_n\}$ is Cauchy, and by completeness, there is $x \in X$ such that $x_n \to x$ in X. By closedness, we have $x \in \overline{B(x_n, r_n)}$, and since $\overline{B(x_n, r_n)} \subset X \setminus C_n$, we have shown that there is $x \in X$ such that $x \notin C_n$ for all n. □

This proof can be slightly modified to get other forms of the Baire theorem.

Exercise 2. Prove the following statements.

a) A complete metric space cannot be written as a countable union of nowhere dense sets.
b) The intersection of a countable collection of open dense subsets of a complete metric space is again dense.

3. The uniform boundedness principle

The uniform boundedness principle was proved in 1927 by Stefan Banach and Hugo Steinhaus, and independently by Hans Hahn. We fist look at a version of this principle for families of continuous functions.

Theorem 3 (Uniform boundedness)**.** *Let X be a complete metric space, and let Y be a normed linear space. Suppose that F is a collection of continuous functions $f : X \to Y$ satisfying*

$$\sup_{f \in F} \|f(x)\| < \infty, \tag{1}$$

for each $x \in X$. Then there is a nonempty open set $B \subset X$ such that

$$\sup_{x \in B} \sup_{f \in F} f(x) < \infty. \tag{2}$$

In other words, pointwise boundedness of continuous functions on a complete metric space implies uniform boundedness on a nonempty open set.

Proof. The sets

$$C_n = \bigcap_{f \in F} \{x \in X : \|f(x)\| \leq n\}, \tag{3}$$

are closed, and $\bigcup_n C_n = X$, so by Baire's theorem at least one of C_n contains an open ball. □

If we apply the above principle to families of continuous linear operators between normed spaces, we get the following corollary, which is called the Banach-Steinhaus theorem.

Theorem 4 (Banach-Steinhaus). *Let X and Y be a Banach and normed spaces, respectively, and let $\mathfrak{A}$ be a collection of bounded linear operators $A : X \to Y$ such that*

$$\sup_{A\in\mathfrak{A}} \|Ax\| < \infty, \tag{4}$$

for each $x \in X$. Then we have

$$\sup_{A\in\mathfrak{A}} \|A\| < \infty. \tag{5}$$

In other words, pointwise boundedness of linear operators on a Banach space implies uniform boundedness.

Proof. An application of Theorem 3 to the collection $\mathfrak{A}$ yields the existence of a ball $B_\varepsilon(z)$ with $\varepsilon > 0$ (and $z \in X$) such that

$$\alpha := \sup_{x\in B_\varepsilon(z)} \left(\sup_{A\in\mathfrak{A}} \|Ax\|\right) < \infty. \tag{6}$$

Now if $x \in B_\varepsilon(0)$, then we can bound $\|Ax\|$ by using the triangle inequality as

$$\|Ax\| = \|A(z + x) - Az\| \leq \|A(z + x)\| + \|Az\| \leq 2\alpha. \tag{7}$$

Finally, for arbitrary $x \in X$, a simple scaling argument gives

$$\|Ax\| = \frac{2\|x\|}{\varepsilon} \left\|A\left(\frac{\varepsilon x}{2\|x\|}\right)\right\| \leq \frac{4\alpha}{\varepsilon}\|x\|, \tag{8}$$

meaning that $\|A\| \leq 4\alpha/\varepsilon$ independent of $A \in \mathfrak{A}$. □

Exercise 3. Show that the preceding theorem would not be true if X was not complete.

The following corollary embodies a typical application.

Corollary 5. *Let X and Y be a Banach and normed spaces, respectively, and let $\{A_n\}$ be a sequence of bounded linear operators between X and Y. Suppose that for each $x \in X$, there exists $y = y(x) \in Y$ such that*

$$A_n x \to y \qquad \text{in} \quad Y, \quad \text{as} \quad n \to \infty. \tag{9}$$

Then we have

$$\sup_n \|A_n\| < \infty. \tag{10}$$

In other words, pointwise convergence implies uniform boundedness.

The proof is straightforward because the convergence $A_n x \to y$ (as $n \to \infty$) implies $\sup_n \|A_n x\| < \infty$ for each x. This corollary can be used to give a quick proof of the fact that there exists a continuous function whose Fourier series diverges in the uniform norm, by evaluating the norm (which grows like $\log n$) of the partial summation operator as an operator acting in the space of continuous functions.

4. The open mapping theorem

The open mapping theorem is one of the cornerstones of linear functional analysis. It is sometimes called the Banach-Schauder theorem, after Stefan Banach and Juliusz Schauder.

A mapping is called *open* if it sends open sets to open sets. In view of Lemma 1, observe that openness is "continuity in the wrong direction", in the sense that if exists, the inverse of a continuous mapping is open. To get some rough feeling of what open mappings do, thinking of maps between finite dimensional spaces, if a set is "expanding in all possible directions", then the image of this process under an open mapping will look similar, "expanding in all possible directions". As an example of this behavior, if a linear operator between normed spaces is open, it must be surjective. Indeed, under an open linear mapping $T : X \to Y$, an open neighborhood of $0 \in X$ goes to an open neighborhood U of $0 \in Y$, and for any $y \in Y$ there is $\alpha \neq 0$ such that $\alpha y \in U$. The following theorem says that the converse is also true when the two spaces are complete.

Theorem 6 (Open mapping)**.** *Let $A : X \to Y$ be a bounded linear operator between two Banach spaces. Then A is surjective if and only if it is open.*

Proof. Suppose that A is surjectve. This implies $Y = \bigcup_{n\in\mathbb{N}} \overline{A(B_n(0))}$, and hence by Baire, there is a nonempty ball $B_\delta(y) \subset Y$ and $n \geq 1$ such that $B_\delta(y) \subset \overline{A(B_n(0))}$. By choosing $x \in X$ such that $y = Ax$, and $m > 0$ so large that $B_m(x) \supset B_n(0)$, we can guarantee $B_\delta(y) \subset \overline{A(B_m(x))}$. By linearity, with $\alpha = \delta/m$ we have $B_{\alpha r}(Ax) \subset \overline{A(B_r(x))}$ for all $x \in X$ and all $r > 0$. If the inclusion did not have the closure in the right hand side, this statement is exactly what we wanted. The proof is completed by the following lemma. □

Lemma 7. *Let X and Y be Banach and normed spaces, respectively, and let $A : X \to Y$ be a linear operator. Suppose that there is some $\alpha > 0$ such that $B_{\alpha r}(Ax) \subset \overline{A(B_r(x))}$ for all $x \in X$ and all $r > 0$. Then A is open.*

Proof. Let $z \in B_{\alpha r}(Ax)$, and fix some $\varepsilon \in (0, 1)$. Then there is $x_0 \in B_r(x)$ such that $\|z - Ax_0\| < \alpha\varepsilon$, which implies that $z \in B_{\alpha\varepsilon}(Ax_0) \subset \overline{A(B_\varepsilon(x_0))}$. This means that there is $x_1 \in B_\varepsilon(x_0)$ such that $\|z - Ax_1\| < \alpha\varepsilon^2$. By iterating, we get a sequence $\{x_n\}$ in X satisfying $\|x_n - x_{n-1}\| < \varepsilon^n$ and $\|z - Ax_n\| < \alpha\varepsilon^n$ for $n \in \mathbb{N}$. From the latter property we have $z = Ax_*$ with $x_* = \lim x_n$, and from the former we infer $\|x_* - x\| < r_* := r + \varepsilon/(1 - \varepsilon)$, meaning that $B_{\alpha r}(Ax) \subset A(B_{r_*}(x))$. If we squeeze this argument we can get the result with $r_* = r$, but what we have is already sufficient for establishing the lemma. □

Exercise 4. Improve the above proof to get $r_* = r$.

In view of Lemma 1, the open mapping theorem implies that if the inverse A^{-1} exists, then it must be continuous. Since continuity is equivalent to boundedness for linear operators on normed spaces, we obtain the following result, which is sometimes called Banach's bounded inverse theorem.

Corollary 8 (Bounded inverse)**.** *Let $A : X \to Y$ be an invertible bounded linear operator between two Banach spaces. Then the inverse $A^{-1} : Y \to X$ is bounded.*

A map $T : X \to Y$ can be identified with its *graph*

$$\mathrm{graph}(T) = \{(x, Tx) : x \in X\} \subset X \times Y. \tag{11}$$

Suppose that (X, ρ) and (Y, σ) are complete metric spaces, and equip $X \times Y$ with the metric $\rho + \sigma$. If T is continuous, obviously $\mathrm{graph}(T)$ is closed, since $(x_n, Tx_n) \to (x, y)$ in $X \times Y$ implies $y = Tx$. In the linear world, the converse is also true.

Theorem 9 (Closed graph)**.** *Let $A : X \to Y$ be a linear operator between two Banach spaces. Then A is bounded if and only if its graph is closed.*

Proof. Suppose that $\mathrm{graph}(A)$ is closed in $X \times Y$, i.e., that it is a Banach space. Define the two operators $\pi_1 : \mathrm{graph}(A) \to X$ and $\pi_2 : \mathrm{graph}(A) \to Y$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, respectively. It is clear that the both operators are bounded linear, and that π_1 is invertible. Since we have $A = \pi_2\pi_1^{-1}$, the claim follows from an application of Corollary 8 to π_1. □