MATH 581 ASSIGNMENT 5

DUE WEDNESDAY APRIL 9

1. For each of the following functions, determine if it is a tempered distribution, and if so compute its Fourier transform.
   a) $x \sin x$,
   b) $\frac{1}{x} \sin x$,
   c) $e^{i|x|^2}$,
   d) $x\vartheta(x)$, where ϑ is the Heaviside step function,
2. For $t > 0$, let $\varkappa_t : \mathbb{R}^n \to \mathbb{R}^n$ be the dilation operator $\varkappa_t(x) = tx$. We call a function f defined either on $\Omega = \mathbb{R}^n$ or on $\Omega = \mathbb{R}^n \setminus \{0\}$ *homogeneous of degree* $s \in \mathbb{C}$ if $\varkappa_t^* f = t^s f$ for all $t > 0$, that is, $f(tx) = t^s f(x)$ for all $t > 0$ and $x \in \Omega$.
   a) Extend the dilation operator $\varkappa_t$ to distributions on Ω, and give a natural definition of homogeneous distributions.
   b) Show that differentiation of distributions preserves homogeneity.
   c) Derive a formula for $\widehat{u \circ A}$, where A is an $n \times n$ invertible matrix.
   d) Show that the Fourier transform of a homogeneous distribution of degree s is a homogeneous distribution of degree $-s - n$.
   e) From homogeneity considerations, compute the inverse Fourier transform of $|\xi|^{-2}$ up to a constant factor. Then determine this constant factor. Take the space dimension to be $n \geq 3$.
3. *a)* There are (at least) two ways to define the Fourier transform on $L^2(\mathbb{R}^n)$.
   - Extend the Fourier transform from $\mathscr{S}$ to L^2 by using the density of $\mathscr{S}$ in L^2 (as well as the Plancherel bound).
   - First define the Fourier transform on $\mathscr{S}'$ by duality, and then restrict it to L^2.

   Show that these two approaches are consistent with each other.
   b) Show that the Fourier transform acting on L^1 is not onto $\mathscr{C}_0$.
4. *a)* Give an example of $u \in \mathscr{C}(\mathbb{R}^n)$ such that $\varphi \mapsto \int u\varphi$ is a tempered distribution and that there is no polynomial p satisfying $|u(x)| \leq |p(x)|$ for all $x \in \mathbb{R}^n$.
   b) Show that the Radon measure
   $$\varphi \mapsto \sum_{k=1}^{\infty} a_k \varphi(k),$$
   is in $\mathscr{S}'(\mathbb{R})$ if and only if there are constants $m \geq 0$ and $c > 0$ such that
   $$|a_k| \leq ck^m, \qquad \text{for all } k.$$

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5. a) Prove that if p is a polynomial with no real zeroes, then there are constants $c > 0$ and m such that $|p(\xi)| \geq c(1+|\xi|)^m$ for all $\xi \in \mathbb{R}^n$. Operators $p(D)$ with p satisfying this condition are called *strictly elliptic*.
   b) Show that if $p(D)$ strictly elliptic, then the equation $p(D)u = f$ has a solution for each $f \in \mathscr{S}'$.
6. Prove the following.
   a) The *Paley-Wiener-Schwartz theorem*: Let $K \subset \mathbb{R}^n$ be a compact convex set, and let $\psi \in \mathscr{S}'$. Then a necessary and sufficient condition for ψ to be the Fourier transform of a distribution supported in K is that ψ is entire and satisfies the growth estimate

$$|\psi(\zeta)| \leq C(1+|\zeta|)^N e^{I_K(\eta)}, \qquad \zeta = \xi + i\eta \in \mathbb{C}^n,$$

   with some constants C and N. Hence the Fourier-Laplace transform of a compactly supported distribution is an entire function of exponential type. Recall that the indicator function I_K of K is defined as

$$I_K(\eta) = \sup_{x \in K} \eta \cdot x.$$

   b) If the set of real zeroes of p is bounded, then every tempered distribution solution of $p(D)u = 0$ is an entire function of exponential type.
   c) If $p(D)$ is a nontrivial differential operator and $u \in \mathscr{E}'$ satisfies $p(D)u = 0$ then $u = 0$.
7. Let p be a nonzero polynomial. Show the following.
   a) The equation $p(D)u = f$ has at least one smooth solution for every $f \in \mathscr{D}$.
   b) If all solutions of $p(D)u = 0$ are smooth, then $p(D)$ is hypoelliptic.
   c) If $p(D)$ admits a fundamental solution that is smooth outside some ball of finite radius (centred at the origin), then $p(D)$ is hypoelliptic.
8. Recall that by Hörmander's theorem, $p(D)$ is hypoelliptic if and only if for any $\eta \in \mathbb{R}^n$ one has $p(\xi + i\eta) \neq 0$ for all sufficiently large $\xi \in \mathbb{R}$.
   a) Construct a non-hypoelliptic polynomial p in dimension $n > 1$ such that $|p(\xi)| \to \infty$ as $|\xi| \to \infty$ for $\xi \in \mathbb{R}^n$.
   b) For any given $c > 0$, construct a non-hypoelliptic polynomial p in dimension $n > 1$ such that $|p(\xi + i\eta)| \to \infty$ uniformly in $\{|\eta| \leq c\}$ as $|\xi| \to \infty$ for $\xi \in \mathbb{R}^n$.