MATH 581 ASSIGNMENT 3

DUE MONDAY MARCH 10

1. Let $\varphi \in \mathscr{D}(\mathbb{R})$, $\varphi \neq 0$, and $\varphi(0) = 0$. In each of the following cases, decide if $\varphi_j \to 0$ as $j \to \infty$ in $\mathscr{D}(\mathbb{R})$. Does it hold $\varphi_j \to 0$ pointwise or uniformly?
   a) $\varphi_j(x) = j^{-1}\varphi(x - j)$;
   b) $\varphi_j(x) = j^{-n}\varphi(jx)$, where $n > 0$ is an integer.
2. *a)* Show that a linear map $f : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega')$ is continuous iff for each compact set $K \subset \Omega$ there exists a compact set $K' \subset \Omega'$ such that $f : \mathscr{D}(K) \to \mathscr{D}(K')$ is continuous.
   b) Show that $L : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ is continuous, where L is a linear differential operator with smooth coefficients.
3. Prove that the space $\mathscr{D}^m(\Omega)$ with its inductive limit topology is sequentially complete.
4. Show that any closed bounded subset of $\mathscr{E}(\Omega)$ is compact in $\mathscr{E}(\Omega)$.
5. In each case, show that f defines a distribution on $\mathbb{R}^2$, and find its order.
   a) $f(\varphi) = \int_{\mathbb{R}^2} |x|^{-1}e^{|x|^2}\varphi(x)\mathrm{d}x$;
   b) $f(\varphi) = \int_{\mathbb{R}} \varphi(s, 0)\mathrm{d}s$;
   c) $f(\varphi) = \int_0^1 \partial_1\varphi(0, s)\mathrm{d}s$.
6. Compute the following derivatives in the sense of distributions.
   a) $\partial_x|x|$;
   b) $\partial_x \mathrm{sign}\, x$ ($\mathrm{sign}\, x = 0$ if $x = 0$ and $\mathrm{sign}\, x = x/|x|$ otherwise);
   c) $\partial_x \log|x|$;
   d) $\partial_2 f$, where $f \in \mathscr{D}'(\mathbb{R}^2)$ is the distribution from *b)* of the previous exercise.
7. Prove the following.
   a) $\partial_j(au) = (\partial_j a)u + a(\partial_j u)$ for $a \in C^\infty(\Omega)$ and $u \in \mathscr{D}'(\Omega)$.
   b) $\partial_j\partial_k u = \partial_k\partial_j u$ for $u \in \mathscr{D}'(\Omega)$.
   c) If $u_k \to u$ in $\mathscr{D}'(\Omega)$ then $\partial_j u_k \to \partial_j u$ in $\mathscr{D}'(\Omega)$.
   d) There is no distribution on $\mathbb{R}$ such that its restriction to $\mathbb{R} \setminus \{0\}$ is $e^{1/x}$.
8. If u is the characteristic function of the unit ball in $\mathbb{R}^n$, compute $x \cdot \nabla u$.
9. Let $\omega \subset \mathbb{R}$ be an open interval, and $u \in \mathscr{D}'(\omega)$. Let $0 \leq k \leq \infty$.
   a) Show that if $u' = 0$ then u is a constant function.
   b) Show that if $u' = f$ with $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.
   c) Show that if $u' + au = f$ with $a \in C^\infty(\omega)$ and $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.
10. Find the limits $n \to \infty$ of the following sequences in $\mathscr{D}'(\mathbb{R})$.
   a) $n\phi(nx)$, where ϕ is a nonnegative continuous function whose integral over $\mathbb{R}$ is finite.
   b) $n^k \sin nx$, where $k > 0$ is a constant.
   c) $x^{-1} \sin nx$.

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11. *a)* Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $u \in \mathscr{E}'^m(\Omega)$. Show that there exists $g \in C(\mathbb{R}^n)$ such that $u = \partial_1^{m+2} \dots \partial_n^{m+2} g$ in Ω.
    b) Let $u \in \mathscr{D}'(\Omega)$. Show that for any open $\omega \subset \Omega$ with $\overline{\omega} \Subset \Omega$, there exists $g \in C(\mathbb{R}^n)$ and a multi-index α such that $u = \partial^\alpha g$ in ω.
12. Show that for any given $\varepsilon > 0$, there exists a finite sequence $f_0, f_1, \dots, f_n$ of continuous functions on $\mathbb{R}$, such that $\operatorname{supp} f_k \subset [-\varepsilon, \varepsilon]$ for each k and

$$\delta = f_0 + f_1' + \dots + f_n^{(n)},$$

where $\delta \in \mathscr{D}'(\mathbb{R})$ is the Dirac mass concentrated at the origin. Prove that there is no continuous function f with compact support such that $\delta = f^{(n)}$ for some n.