MATH 581 ASSIGNMENT 2

DUE WEDNESDAY FEBRUARY 12

1. For each of the following cases, determine the characteristic cones and characteristic surfaces.
   a) Wave equation with wave speed $c > 0$: $u_{xx} + u_{yy} = c^{-2} u_{tt}$.
   b) Tricomi-type equation: $u_{xx} + y u_{yy} = 0$.
   c) Ultrahyperbolic "wave" equation: $u_{xx} + u_{yy} = u_{zz} + u_{tt}$.
2. Prove that if $\beta \in \mathbb{R}$ and $u \in C^1(\mathbb{R}^2)$ is a solution of $u_t + \beta u_x = 0$, then
$$\{(x,t) : u \in C^k \text{ on a neighbourhood of } (x,t)\},$$
is a union of lines.
3. Consider the Laplace equation $\Delta u = 0$ on the unit disk, given in polar coordinates by $\mathbb{D} = \{(r,\theta) : r < 1\}$. Specify the Cauchy data
$$u(1,\theta) = f(\theta), \qquad \partial_r u(1,\theta) = g(\theta),$$
where f and g are 2π-periodic real analytic functions. Then show that a real analytic solution exists in a neighbourhood of the circle $\partial\mathbb{D}$. Investigate what happens to the solution as $r \to 0$ and $r \to \infty$, if f and g are of the form
$$a_0 + \sum_{n=1}^{m} a_n \cos n\theta + b_n \sin n\theta,$$
i.e., trigonometric polynomials.
4. Consider the wave equation
$$u_{tt} - u_{xx} = f,$$
with the initial data
$$u(x, \alpha x) = \phi(x) \quad \text{for} \quad x < 0, \qquad \text{and} \qquad u(x,x) = \psi(x) \quad \text{for} \quad x \geq 0,$$
where $\alpha \neq 1$ is a constant, and ϕ and ψ are real analytic functions in a neighbourhood of $0 \in \mathbb{R}$. Note that we are specifying the initial condition on the union of two rays, one of which is characteristic, and the other may or may not be characteristic, depending on α. Supposing that f is real analytic in a neighbourhood of $0 \in \mathbb{R}^2$, investigate if and when the problem is locally (analytically) solvable near $0 \in \mathbb{R}^2$. Do we need to impose compatibility conditions on the data ϕ and ψ?
5. Let p be a nontrivial polynomial of n variables, and let f be a real analytic function in a neighbourhood of $0 \in \mathbb{R}^n$.
   a) Prove that the set $\{\xi \in \mathbb{R}^n : p(\xi) = 0\}$ is closed and of measure zero.

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b) Show that there is a neighbourhood of $0 \in \mathbb{R}^n$, on which the equation $p(\partial)u = f$ has a solution. Supposing that $p(\xi) = \sum_\alpha a_\alpha \xi^\alpha$, here the operator $p(\partial)$ is given by
$$p(\partial) = \textstyle\sum_\alpha a_\alpha \partial^\alpha.$$

c) Extend this local solvability result to linear operators with analytic coefficients. That is, assuming that $\{a_\alpha\}$ is a finite collection of real analytic functions in a neighbourhood of $0 \in \mathbb{R}^n$, with the property that $p(\xi) = \sum_\alpha a_\alpha(0)\xi^\alpha$ is a nontrivial polynomial, show that the equation
$$\textstyle\sum_\alpha a_\alpha \partial^\alpha u = f,$$
has a solution on a neighbourhood of $0 \in \mathbb{R}^n$.

6. Let p be a nontrivial polynomial of n variables, and let $H \subset \mathbb{R}^n$ be a (closed) half-space.
   a) Show that if $u \in C^\infty(\mathbb{R}^n)$ satisfies $p(\partial)u = 0$ in $\mathbb{R}^n$ and $\operatorname{supp} u \subset H$, and if the boundary of H is noncharacteristic for the constant coefficient operator $p(\partial)$, then $u \equiv 0$. Provide a counterexample when ∂H is characteristic and p is a nonconstant homogeneous polynomial.
   b) Show that if we require that u is compactly supported, then the noncharacteristic condition on ∂H can be dropped, i.e., prove that if $u \in C_c^\infty(\mathbb{R}^n)$ satisfies $p(\partial)u = 0$ in $\mathbb{R}^n$ then $u \equiv 0$. Imply that if $u \in C_c^\infty(\mathbb{R}^n)$ then $\operatorname{supp} u$ is contained in the convex hull of $\operatorname{supp} p(\partial)u$.
7. Let u be a C^2 solution of the n-dimensional wave equation $\partial_t^2 u - \Delta u = 0$, and assume that u and all its first derivatives vanish on the line segment $\{(0,t) \in \mathbb{R}^{n+1} : 0 < t < T\}$. By using Holmgren's theorem, determine the region of $\mathbb{R}^{n+1}$ where u must vanish.