MATH 581 ASSIGNMENT 1

DUE WEDNESDAY JANUARY 29

1. Consider the function $v(x,t) = \frac{x}{t} E(x,t)$ for $x \in \mathbb{R}$ and $t > 0$, where
$$E(x,t) = \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{|x|^2}{4t}} \qquad (x \in \mathbb{R},\ t > 0),$$
is the heat kernel of $\mathbb{R}$. Show that $\partial_t v = \Delta v$ in $\mathbb{R} \times (0,\infty)$, and that $v(x,t) \to 0$ as $t \to 0^+$ for each fixed $x \in \mathbb{R}$. How do we reconcile this with Tychonov's uniqueness theorem?
2. With $\Omega \subset \mathbb{R}^n$ a bounded smooth domain, consider the initial-boundary value problem
$$\begin{cases} \partial_t u = \Delta u + au & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u = g & \text{on} \quad \Omega \times \{0\}, \end{cases}$$
where $g \in C(\bar{\Omega})$ and $a \in L^\infty(\Omega \times (0,\infty))$ are given functions. Assume the existence of a solution u in the class $C^2(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$.
   a) Show that the solution is unique in the same class.
   b) Assuming $a \equiv 0$, show that
$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq (4\pi t)^{-\frac{n}{2}} \|g\|_{L^1(\Omega)}, \qquad \text{for all} \quad t > 0.$$
   c) Show that there exists $c > 0$ with the property that if $\|a\|_\infty \leq c$ then the L^2-norm of $u(\cdot,t)$ decays exponentially in time.
   d) Under some smallness condition on a, can you establish an exponential decay in stronger norms, such as H^k or L^∞?
3. With $\Omega \subset \mathbb{R}^n$ a bounded smooth domain, consider the initial-boundary value problem
$$\begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u = g & \text{on} \quad \Omega \times \{0\}, \end{cases}$$
where ∂_ν is the outward normal derivative at $\partial\Omega$, and $g \in C(\bar{\Omega})$ is a given function. Assume the existence of a solution u in the class $C^2(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$.
   a) Prove the maximum principle:
$$\sup_{\Omega \times (0,\infty)} u \leq \sup_{\Omega} g, \qquad \text{and} \qquad \inf_{\Omega \times (0,\infty)} u \geq \inf_{\Omega} g.$$
   b) Show that the solution is unique in the same class.

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c) Show that $u(\cdot,t)$ converges uniformly to a constant as $t\to\infty$. What can you say about the *rate* of convergence?

4. (Duhamel's principle) Let $g\in C_b(\mathbb{R}^n)$, and let $f\in C^1(\mathbb{R}^n\times(0,\infty))$ be a function satisfying
$$\sup_{\mathbb{R}^n\times(0,\beta)}|f|<\infty, \qquad \text{and} \qquad \sup_{\mathbb{R}^n\times(\alpha,\beta)}|\nabla_x f|<\infty,$$
for each $0<\alpha<\beta<\infty$, where $C_b(\mathbb{R}^n)$ is the space of bounded continuous functions on $\mathbb{R}^n$, and $\nabla_x=(\partial_{x_1},\dots,\partial_{x_n})$ is the "spatial gradient". Prove that
$$u(x,t)=\int_{\mathbb{R}^n}E(x-y,t)g(y)\,\mathrm{d}y+\int_0^t\int_{\mathbb{R}^n}E(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$
satisfies the inhomogeneous heat equation
$$\partial_t u-\Delta u=f, \qquad \text{in} \quad \mathbb{R}^n\times(0,\infty),$$
with $u(\cdot,t)\to g$ locally uniformly in $\mathbb{R}^n$ as $t\to 0^+$. Here
$$E(x,t)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}} \qquad (x\in\mathbb{R}^n,\ t>0),$$
is the heat kernel of $\mathbb{R}^n$.