ELEMENTS OF THE ABSTRACT THEORY OF HILBERT SPACE SCATTERING

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1. Introduction

Scattering theory is the study of a physical system on time scales that are large compared to the scale of interactions taking place within the system. Often, one has in mind two sets of dynamics acting on a sytem - a free dynamics which is easy to solve, and an interacting dynamics which is more difficult. An example arising in quantum mechanics is the free dynamics of a particle induced by the Laplacian $-\Delta$, and then the interacting dynamics induced by a central potential, $-\Delta + V$. Scattering theory gives tools to examine the large time behaviour of interacting sytems using knowledge of free systems. In a sense, scattering theory is a type of perturbation theory.

In this paper we will be interested in some fundamentals of Hilbert space scattering theory. In the first section we will recall some of the basic definitions and concepts associated with the theory of linear operators on Hilbert spaces. In the second section we will examine two basic elements of Hilbert space scattering, Cook's method and the Kato-Birman theory.

2. Linear Operators on Hilbert Spaces

We recall the definitions and elementary theory of linear operators on Hilbert space for reader convenience and to fix our notation. We follow closely [J], and refer the reader to [J] for proofs. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. The inner product of $\mathcal{H}_i$ is denoted by $\langle \cdot, \cdot \rangle_i$. When the meaning is clear (i.e., there is only one Hilbert space under consideration) we will denote the inner product by $\langle \cdot, \cdot \rangle$ (i.e., drop the suffix). In all cases, the inner product is linear wrt the second variable. Throughout this paper we assume that all Hilbert spaces are separable.

A **linear operator** A from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a linear map from a distinguished subspace $D(A)$ of $\mathcal{H}_1$ to $\mathcal{H}_2$. $D(A)$ is called the **domain** of A, and A is called **densely defined** if $D(A)$ is dense in $\mathcal{H}_1$. If A and B are linear operators, then $A + B$ is defined on $D(A + B) = D(A) \cap D(B)$. Similarly, AB is defined on

$$D(AB) = \{\psi : \psi \in D(B), B\psi \in D(A)\}.$$

B is called an **extension** of A if $D(A) \subseteq D(B)$ and $A\psi = B\psi$ for $\psi \in D(A)$. If B extends A we write $A \subseteq B$.

An operator A is called **bounded** if $D(A) = \mathcal{H}$ and

$$\|A\| := \sup_{\|\psi\|=1} \|A\psi\| < \infty.$$

The set of bounded operators between Hilbert spaces is denoted $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and we also define $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$. If A is densely defined and there is a constant C so that $\|A\psi\| \leq C \|\psi\|$ holds for every $\psi \in D(A)$, then A has a unique extension to an element of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

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The **graph** of A is defined as

$$\Gamma(A) := \{(\psi, A\psi) : \psi \in D(A)\} \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2.$$

An operator is called **closed** if $\Gamma(A)$ is a closed subset of $\mathcal{H}_1 \oplus \mathcal{H}_2$ in the topologly induced by the norm $(\psi, \varphi) \to \langle \psi, \psi \rangle_{\mathcal{H}_1} + \langle \varphi, \varphi \rangle_{\mathcal{H}_2}$.

An operator is called **closeable** if it has a closed extension. If A is closeable, its smallest-closed extension is called its closure and is denoted by $\bar{A}$. It is an elementary fact [RS1] that A is closeable iff $\overline{\Gamma(A)}$ is the graph of a linear operator. In this case, $\Gamma(\bar{A}) = \overline{\Gamma(A)}$.

Let A be a closed operator. A subset $D \subseteq D(A)$ is called a core for A if $\overline{A \restriction D} = A$.

2.1. **Adjoints.** We now turn to the task of defining the **adjoint**, A^* of a densely defined linear operator A. In this section and the remainder of this report, we will only consider linear maps from $\mathcal{H}$ to itself. The set of all $\phi \in \mathcal{H}$ for which there exists a $\psi \in \mathcal{H}$ so that the equality

$$\langle A\varphi, \phi \rangle = \langle \varphi, \psi \rangle$$

holds for every $\varphi \in \mathcal{H}$ is defined as $D(A^*)$, and we set $A^*\phi = \psi$. It is easy to see that A^* is a well-defined linear operator. Moreover, we have

Proposition 1. *Let A be a densely defined linear operator. Then the adjoint A^* is closed, and A is closeable iff $D(A^*)$ is dense, and in this case $\bar{A} = A^{**}$. Finally, if A is closeable, then $\bar{A}^* = A^*$.*

2.2. **The spectrum.** Let A be a closed densely defined operator. The **resolvent** set of A is denoted by $\rho(A)$ and is the set of all $z \in \mathbb{C}$ such that

$$A - z : D(A) \to \mathcal{H}$$

is a bijection. By the closed graph theorem, $(A - z)^{-1} \in \mathcal{B}(\mathcal{H})$. The **spectrum** of A, $\mathrm{sp}(A)$ is defined by

$$\mathrm{sp}(A) := \mathbb{C} \backslash \rho(A).$$

2.3. **Self-adjoint operators.** Let A be a densely defined linear operator on a Hilbert space $\mathcal{H}$. A is called **symmetric** if for every ϕ and ψ in $D(A)$,

$$\langle A\phi, \psi \rangle = \langle \phi, A\psi \rangle. \tag{1}$$

Equivalently, A is symmetric if $A \subseteq A^*$. Any symmetric operator is closeable, and $\bar{A} \subseteq A^*$. A densely defined operator A is called **self-adjoint** if $A = A^*$. A is self-adjoint iff A is symmetric and $D(A) = D(A^*)$. We have the following basic result about self-adjointness:

Proposition 2. *Let A be a symmetric operator on $\mathcal{H}$. Then TFAE:*

(i) *A is self-adjoint.*
(ii) *A is closed and* $\ker(A^* \pm \mathrm{i}) = \{0\}$.
(iii) $\mathrm{ran}(A \pm \mathrm{i}) = \mathcal{H}$.

We say that a symmetric operator A is **essentially self-adjoint** if $\bar{A}$ is self-adjoint.

Proposition 3. *Let A be a symmetric operator on $\mathcal{H}$. The TFAE:*

(i) *A is essentially self-adjoint.*
(ii) $\ker(A^* \pm \mathrm{i}) = \{0\}$.
(iii) $\mathrm{ran}(A \pm i)$ *is dense in $\mathcal{H}$.*

2.4. **The spectral theorem.** In this section we state the basic structural result for self-adjoint operators. Let $\psi \in \mathcal{H}$ and A a self-adjoint operator. The **cyclic subspace** generated by ψ and A is the closure of the linear span of the vectors

$$\{(A-z)^{-1}\psi : z \in \mathbb{C}\backslash\mathbb{R}\}$$

We have:

Theorem 1 (Decomposition theorem)**.** *Let A be a self-adjoint operator. Then there is a set of orthonormal vectors $\{\phi_n\}$ so that*

$$\mathcal{H} = \oplus_n \mathcal{H}_n, \qquad A = \oplus_n A \restriction_{\mathcal{H}_n},$$

where $\mathcal{H}_n$ is the cyclic subspace for ϕ_n and A. Each $\mathcal{H}_n$ is invariant under A and the subspaces $\mathcal{H}_n$ are mutually orthogonally closed.

The set $\{\phi_n\}$ and subspaces $\{\mathcal{H}_n\}_n$ is called a **cyclic decomposition** for A.

A vector $\psi \in \mathcal{H}$ is called **cyclic** for A if the cyclic subspace generated by ψ and A is all of $\mathcal{H}$. We have,

Theorem 2 (Spectral theorem, cyclic case)**.** *Let A be a self-adjoint operator, and $\psi \in \mathcal{H}$. Then there is a unique Borel measure, denoted $\mathrm{d}\mu_\psi$ s.t. $\mu_\psi(\mathbb{R}) = \|\psi\|^2$ and*

$$\langle \psi, (A-z)^{-1}\psi \rangle = \int_{\mathbb{R}} \frac{1}{x-z} \mathrm{d}\mu_\psi(x)$$

holds for every $z \in \mathbb{C}\backslash\mathbb{R}$. Moreover, if ψ is cyclic for A, then there is a unitary operator U so that A is unitarily equivalent to an operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$.

The two preceding theorems imply

Theorem 3 (Spectral theorem, general case)**.** *Let A be a self-adjoint operator and $\{\psi_n\}_n$ and $\{\mathcal{H}_n\}_n$ be a cyclic decomposition for A. Then there is a unitary operator U,*

$$U : \mathcal{H} = \oplus_n \mathcal{H}_n \to \oplus_n L^2(\mathbb{R}, \mathrm{d}\mu_{\psi_n})$$

so that A acts by multiplication by x on each $L^2(\mathbb{R}, d\mu_{\psi_n})$. Furthermore,

$$\mathrm{sp}(A) = \overline{\bigcup_n \mathrm{supp}\ \mu_{\psi_n}}$$

Let A be a self-adjoint operator and the orthonormal family $\{\psi_n\}_{n\in\Gamma}$ be as in the above theorem. We define the measure space M and sigma-algebra $\mathcal{F}$ as follows. For every $n \in \Gamma$, let $\mathbb{R}_n$ be a copy of $\mathbb{R}$ and let $M = \bigcup_{n\in\Gamma} \mathbb{R}$. Let $\mathcal{F}$ be the collection of sets $F \subseteq M$ such that $F \cap \mathbb{R}_n$ is Borel for every $n \in \Gamma$. For $F \in \mathcal{F}$, let

$$\mu(F) = \sum_{n\in\Gamma} \mu_{\psi_n}(F \cap \mathbb{R}_n).$$

By the above theorem, A is unitarily equivalent to an operator of multiplication on $L^2(M, \mathrm{d}\mu)$. Note that $\mathrm{sp}(A) = \mathrm{supp}\,\mu$. Denote by U this unitary operator. Recall that any Borel measure ν on $\mathbb{R}$ has the decomposition

$$\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{sc}} + \nu_{\mathrm{pp}}$$

where ν_{ac} is absolutely continuous wrt to Lebesgue measure, ν_{pp} is an atomic measure (recall that an atom of a measure is a singleton $\{x\}$ s.t. $\nu(\{x\}) > 0$ - an atomic measure is a measure consisting only of atoms), and ν_{sc} is supported on a set of Lebesgue measure 0 but has no atoms. $\nu_{\mathrm{ac/sc/pp}}$ is called the absolutely continuous/singular continuous/pure point part of ν.

Let,

$$\mu_{\text{ac/sc/pp}} = \sum_{n \in \Gamma} \mu_{\psi_n, \text{ac/sc/pp}}$$

Then, $L^2(M, \mathrm{d}\mu) = L^2(M, \mathrm{d}\mu_{\text{ac}}) \oplus L^2(M, \mathrm{d}\mu_{\text{sc}}) \oplus L^2(M, \mathrm{d}\mu_{\text{pp}})$ and we define

$$\mathcal{H}_{\text{ac/sc/pp}} = U^{-1} L^2(M, \mathrm{d}\mu_{\text{ac/sc/pp}}).$$

These subspaces are invariant under A and are called the absolutely continuous/singular continuous/pure point spectral subspaces for A. The projection onto this subspace is denoted by $P_{\text{ac/sc/pp}}(A)$. Finally, we set

$$\text{sp}_{\text{ac/sc/pp}}(A) = \text{sp}(A \upharpoonright \mathcal{H}_{\text{ac/sc/pp}}) = \overline{\bigcup_{n \in \Gamma} \text{supp}}\ \mu_{\psi_n, \text{ac/sc/pp}}$$

2.5. **Stone's theorem.** We also require the following theorem which provides a 1-to-1 correspondence between strongly continuous one-parameter unitary groups and self-adjoint operators on a Hilbert space.

Theorem 4 (Stone's theorem)**.** *Let $U(t)$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathcal{H}$. That is, $\{U(t)\}_{t \in \mathbb{R}}$ is a family of unitary operators on $\mathcal{H}$ satisfying*

(i) *$U(t)U(s) = U(t+s)$, for every $t, s \in \mathbb{R}$.*
(ii) *The map $t \to U(t)$ is strongly continuous.*

Then, there is a self-adjoint operator A satisfying $\mathrm{e}^{\mathrm{i}tA} = U(t)$. Conversely, for any self-adjoint operator A, the set $\{e^{\mathrm{i}tA}\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group. Moreover, $D(A)$ is precisely the vectors ψ for which the limits

$$\lim_{t \to 0} t^{-1}(e^{\mathrm{i}tA}\psi - \psi)$$

exist, and the limit equals $\mathrm{i}A\psi$.

3. Scattering Theory

In this section, we turn to the main focus of this paper. We discuss briefly quantum mechanics and then prove some of the fundamentals of Hilbert space scattering theory. We follow closely [RS3].

A quantum system is described by a Hilbert space of 'states' and a unitary group acting on that Hilbert space which generates the dynamics. Stone's theorem provides a correspondence between dynamics and self-adjoint operators on the Hilbert space. We say that the generator of the dynamics is the Hamiltonian of the system, typically denoted by H.

Scattering theory is typically interested with two sets of dynamics for the same system; a given, "interacting," dynamics, given by some Hamiltonian H, and a free dynamics given by a different Hamiltonian H_0. Typically, the interacting dynamics is difficult to solve analytically, but the free dynamics is "easy" to deal with in some sense, and conserves the momentum of the individual parts of the system. If the difference between the dynamics is "small" in some sort of sense (for example, we shall see later that one sense of "small" is that $H - H_0$ is trace-class) then information about the behaviour of the system under the free dynamics should yield information about the system under the interacting dynamics. In this sense, scattering theory can be viewed as a sort of pertubation theory.

Let H and H_0 be the Hamiltonians generating respectively the interacting and free dynamics on a physical system described by the Hilbert space $\mathcal{H}$. We say that a state φ is asymptotically free in the distant past if there is a state φ_- so that

$$\lim_{t \to -\infty} \left\| e^{-\mathrm{i}H_0 t}\varphi_- - e^{-\mathrm{i}Ht}\varphi \right\| = 0. \tag{2}$$

Such states are those that, when looking far into the past, are comparable to states evolving due to the free dynamics. Note that (2) is equivalent to

$$\lim_{t\to-\infty} \left\| e^{\mathrm{i}Ht}e^{-\mathrm{i}H_0t}\varphi_- - \varphi \right\| = 0,$$

and so the question of determining which states are asymptotically free in the past is reduced to deciding the existence of strong limits. Similarly, we say that a state φ is **asymptotically free** in the distant future if there is a state φ_+ so that (2) holds, but with $t \to \infty$ instead of $t \to -\infty$.

This prompts the following definition: for self-adjoint A and B acting on $\mathcal{H}$, let $P_{\mathrm{ac}}(B)$ denote the projection onto the absolutely continuous subspace of B. We say that the **generalized wave operators** $\Omega^{\pm}(A,B)$ exist if the strong limits

$$\Omega^{\pm}(A,B) = \operatorname*{s-lim}_{t\to\mp\infty} e^{\mathrm{i}At}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B) \tag{3}$$

exist. When the meaning is clear we will write $\Omega^{\pm}$ for $\Omega^{\pm}(A,B)$. When the generalized wave operators exist, we define

$$\mathcal{H}_{\mathrm{in}} = \operatorname{ran}\Omega^+, \qquad \mathcal{H}_{\mathrm{out}} = \operatorname{ran}\Omega^-. \tag{4}$$

For notational convenience, we will sometimes use the convention $\mathcal{H}_+ = \mathcal{H}_{\mathrm{in}}$ and $\mathcal{H}_- = \mathcal{H}_{\mathrm{out}}$.

When the generalized wave operators exist, the states that look asymptotically free in the distant past/future are the elements of $\mathcal{H}_{\mathrm{in}}/\mathcal{H}_{\mathrm{out}}$, and the associated φ_+ is an element of the absolutely continuous subspace of H_0.

The following is a first basic result about wave operators. Recall that a bounded linear operator U on $\mathcal{H}$ is called a **partial isometry** if $\|U\psi\| = \|\psi\|$ for every $\psi \in (\ker U)^{\perp}$. If U is a partial isometry, then $\mathcal{H}$ can be written as $\mathcal{H} = \ker U \oplus (\ker U)^{\perp}$ and $\mathcal{H} = \operatorname{ran} U \oplus (\operatorname{ran} U)^{\perp}$, with U a unitary operator from $(\ker U)^{\perp}$, which is called the **initial subspace** of U, to $\operatorname{ran} U$, which is called the **final subspace** of U.

Proposition 4. *Suppose that the wave operators exist. Then,*

(i) $\Omega^{\pm}$ *are partial isometries with initial subspace* $P_{\mathrm{ac}}(B)\mathcal{H}$ *and final subspaces* $\mathcal{H}_{\pm}$.

(ii) $\mathcal{H}_{\pm}$ *are invariant subspaces for* A *and*

$$\Omega^{\pm}[D(B)] \subseteq D(A), \qquad A\Omega^{\pm}(A,B) = \Omega^{\pm}(A,B)B. \tag{5}$$

(iii) $\mathcal{H}_{\pm} \subseteq \operatorname{ran} P_{\mathrm{ac}}(A)$.

Proof. For (i), note that obviously $(P_{\mathrm{ac}}\mathcal{H})^{\perp} \subseteq \ker\Omega^{\pm}$. OTOH, if $u \in P_{\mathrm{ac}}\mathcal{H} = ((P_{\mathrm{ac}}\mathcal{H})^{\perp})^{\perp}$ (recall that the range of a bounded linear operator is closed) then $\|e^{\mathrm{i}At}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}u\| = \|u\|$ for every t, and so $\|\Omega^{\pm}u\| = \|u\|$. This shows that $(P_{\mathrm{ac}}\mathcal{H})^{\perp} = \ker\Omega^{\pm}$ and that $\Omega^{\pm}$ is a partial isometry. By definition, the final subspace is $\mathcal{H}_{\pm}$.

To prove (ii), note that for any $s \in \mathbb{R}$, clearly $\Omega^{\pm} = e^{\mathrm{i}As}\Omega^{\pm}e^{=\mathrm{i}Bs}$ (since $e^{\mathrm{i}Bt}$ commutes with $P_{\mathrm{ac}}(B)$, which is, e.g., a corollary of the Borel functional calculus for B). Equivalently,

$$e^{-\mathrm{i}As}\Omega^{\pm} = \Omega^{\pm}e^{-\mathrm{i}Bs}. \tag{6}$$

(5) is then a consequence of Stone's theorem and (6). Moreover, (6) shows that $\mathcal{H}_{\pm}$ is an invariant subspace for $e^{\mathrm{i}As}$, and the invariance for A follows by differentiation.

Lastly, (1) and (2) imply that $A \restriction \mathcal{H}_{\pm}$ is unitarily equivalent to $B \restriction P_{\mathrm{ac}}(B)\mathcal{H}$, which proves (iii) (recalling that the ac/pp/sc parts of the spectrum are preserved under unitaries). □

We have also,

Proposition 5 (Chain rule)**.** *If* $\Omega^{\pm}(A,B)$ *and* $\Omega^{\pm}(B,C)$ *exist, then* $\Omega^{\pm}(A,C)$ *exist and*

$$\Omega^{\pm}(A,C) = \Omega^{\pm}(A,B)\Omega^{\pm}(B,C)$$

Proof. By Proposition 4(iii), $\operatorname{ran}\Omega^{\pm}(B,C) \subseteq P_{\mathrm{ac}}(B)$ and so

$$\lim_{t\to\mp\infty} \left\|(1-P_{\mathrm{ac}}(B))e^{\mathrm{i}tB}e^{-\mathrm{i}tC}P_{\mathrm{ac}}(C)\varphi\right\| = 0$$

for any φ. Therefore,

$$\begin{aligned} e^{\mathrm{i}tA}e^{-\mathrm{i}tC}P_{\mathrm{ac}}(C)\varphi &= e^{\mathrm{i}tA}e^{-\mathrm{i}tB}P_{\mathrm{ac}}(B)e^{\mathrm{i}tB}e^{-\mathrm{i}tC}P_{\mathrm{ac}}(C)\varphi \\ &\quad + e^{\mathrm{i}tA}e^{-\mathrm{i}tB}(1-P_{\mathrm{ac}}(B))e^{\mathrm{i}tB}e^{-\mathrm{i}tC}P_{\mathrm{ac}}(C)\varphi \end{aligned}$$

converges to $\Omega^{\pm}(A,B)\Omega^{\pm}(B,C)\varphi$ as $t \to \mp\infty$. □

If a state looks asymptotically free in the past, we would hope that it would look asymptotically free in the future as well, and vice versa. This notion is captured by the following definition: we say that the wave operators are **weakly asymptotically complete** if $\mathcal{H}_{\mathrm{in}} = \mathcal{H}_{\mathrm{out}}$.

Consider a system composed of individual components for which the interaction between the components falls off whenever the pieces move apart. One expects that states of the system will either decay into freely moving clusters (e.g., two particles moving away from each other to infinity) or will remain bound (e.g., two particles orbiting each under due to a mutual attraction) under the action of the dynamics of the system. In quantum mechanics, the bound states are the elements of $P_{\mathrm{pp}}(H)\mathcal{H}$, where $P_{\mathrm{pp}}(H)$ is the projection onto the pure point part of the interacting Hamiltonian H (they are called bound because they are invariant under the dynamics $e^{\mathrm{i}tH}$).

The definition making this physical notion precise is the following: we say that the wave operators $\Omega^{\pm}(A,B)$ are asymptotically complete if $\mathcal{H}_{\mathrm{in}} = \mathcal{H}_{\mathrm{out}} = (P_{\mathrm{pp}}(A))^{\perp}$. Note that if the wave operators are **asymptotically complete** then $\mathcal{H} = \mathcal{H}_{\pm} \oplus P_{\mathrm{pp}}(A)\mathcal{H}$. Asymptotic completeness implies weak asymptotic completeness.

Sometimes the following definition is useful. If the wave operators $\Omega^{\pm}(A,B)$ exist, we say they are **complete** if $\operatorname{ran}\Omega^{+} = \operatorname{ran}\Omega^{-} = \operatorname{ran}P_{\mathrm{ac}}(A)$.

It is clear that asymptotic completeness is equivalent to the pair of statements: $\Omega^{\pm}(A,B)$ are complete, and A has no singular continuous spectra.

The existence of the wave operators can be proven in many cases by a general technique known as Cook's method, which we will be the next topic of discussion. Under a set of more stringent conditions, one can prove the stronger result that the wave operators exist and are complete, using a complex of ideas called the Kato-Birman theory. These results will be discussed second.

We have the following result about completeness:

Proposition 6. *Suppose that the wave operators $\Omega^{\pm}(A,B)$ exist. Then they are complete iff $\Omega^{\pm}(B,A)$ exist.*

Proof. If both $\Omega^{\pm}(A,B)$ and $\Omega^{\pm}(B,A)$, then by Proposition 5,

$$P_{\mathrm{ac}}(A) = \Omega^{\pm}(A,A) = \Omega^{\pm}(A,B)\Omega^{\pm}(B,A),$$

and so

$$P_{\mathrm{ac}}(A)\mathcal{H} \subseteq \operatorname{ran}\Omega^{\pm}(A,B).$$

The reverse inclusion follows from Proposition 4(iii). For the converse statement, suppose that $\Omega^{\pm}(A,B)$ exist and are complete. Let $\varphi \in P_{\mathrm{ac}}(A)\mathcal{H}$ be given. We want to prove the existence of the limit

$$\lim_{t\to\mp\infty} e^{\mathrm{i}tB}e^{-\mathrm{i}tA}\varphi. \tag{7}$$

By the completeness assumption, there is a ψ with $\varphi = \Omega^{\pm}(A,B)\psi$. The vector $P_{\mathrm{ac}}(B)\psi$ is readily seen to be the limit (7). □

The above proposition suggests that proving the completeness of the wave operators is no more difficult than proving existence. However, in applications this is often untrue. Normally one has explicit formulas for one of the operators (i.e., the H_0 describing free dynamics), and proving the existence of $\Omega^{\pm}(H, H_0)$ is easy while proving the existence of $\Omega^{\pm}(H_0, H)$ is difficult.

We now present our first existence result:

Theorem 5 (Cook's method)**.** *Let A and B be self-adjoint operators on $\mathcal{H}$ and suppose there is a set $\mathcal{D} \subseteq D(B) \cap P_{\mathrm{ac}}(B)\mathcal{H}$ which is dense in $P_{\mathrm{ac}}(B)\mathcal{H}$, such that for any $\varphi \in \mathcal{D}$ there is a T_0 satisfying, for $|t| > T_0$,*

$$e^{-\mathrm{i}Bt}\varphi \in D(A) \tag{8}$$

and

$$\int_{T_0}^{\infty} \left\|(B - A)e^{iBt}\varphi\right\| + \left\|(B - A)e^{\mathrm{i}Bt}\right\| \mathrm{d}t < \infty. \tag{9}$$

Then $\Omega^{\pm}(A, B)$ exist.

Remark. Note that $B - A$ makes sense in (9) by the assumption (8).

Proof. Fix some $\varphi \in \mathcal{D}$, and let $f(t) = e^{\mathrm{i}At}e^{-\mathrm{i}Bt}\varphi$. By Stone's theorem together with the assumption that $f(t) \in D(A) \cap D(B)$ for $t > T_0$, we have that $f(t)$ is strongly differentiable for all $t > T_0$. For $t > s > T_0$, we apply the FTC to write,

$$f(t) - f(s) = \int_s^t -\mathrm{i}e^{\mathrm{i}Au}(B - A)e^{-\mathrm{i}Bu}\varphi \mathrm{d}u. \tag{10}$$

We take the norm on both sides of the equation. Then, (9) implies that $f(t)$ is Cauchy, and so the limit exists and equals

$$\lim_{t\to\infty} e^{\mathrm{i}At}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)\varphi.$$

An $\varepsilon/3$ argument extends this to all of $P_{\mathrm{ac}}(B)\mathcal{H}$, and we define $\Omega^{-}(A, B)$ by setting it 0 on $(P_{\mathrm{ac}}(B)\mathcal{H})^{\perp}$. The proof for the existence of $\Omega^{+}(A, B)$ is identical. □

We also have the following generalization, which is applicable when $B - A$ has "local singularities."

Theorem 6 (Kupsch-Sandhas theorem)**.** *Let A and B be self-adjoint and suppose there is a bounded operator χ and a subspace $\mathcal{D} \subseteq D(B) \cap P_{\mathrm{ac}}(B)\mathcal{H}$ that is dense in $P_{\mathrm{ac}}(B)\mathcal{H}$, so that for any $\varphi \in \mathcal{D}$ there is a T_0 so that for $|t| > T_0$,*

$$(1 - \chi)e^{-\mathrm{i}Bt}\varphi \in D(A) \tag{11}$$

and

$$\int_{T_0}^{\infty} \left\|Ce^{-\mathrm{i}Bt}\varphi\right\| + \left\|Ce^{\mathrm{i}Bt}\varphi\right\| \mathrm{d}t < \infty \tag{12}$$

where $C = A(1 - \chi) - (1 - \chi)B$. If also for some n, the operator $\chi(B + \mathrm{i})^{-n}$ is compact and $\mathcal{D} \subseteq D(B^n)$, then $\Omega^{\pm}(A, B)$ exist.

Proof. We prove that $\Omega^{-}(A, B)$ exists. The proof for $\Omega^{+}(A, B)$ is identical. For fixed $\varphi \in \mathcal{D}$, consider the function $f(t) = e^{\mathrm{i}At}(1-\chi)e^{-\mathrm{i}Bt}\varphi$. The same argument as in the proof of Theorem 5 yields that $f(t)$ is Cauchy, and that $\text{s-lim}_{t\to\infty} e^{\mathrm{i}At}(1 - \chi)e^{-\mathrm{i}Bt}$ extends to a bounded linear operator on $P_{\mathrm{ac}}(B)\mathcal{H}$. We have only left to prove that $e^{\mathrm{i}At}\chi e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)\mathcal{H}$ converges strongly to 0. Clearly, it suffices to prove that $\chi e^{-\mathrm{i}Bt}\varphi$ converges to 0 for $\varphi \in \mathcal{D}$. Since $\varphi \in D(B^n)$ we can write

$$\chi e^{-\mathrm{i}Bt}\varphi = \chi e^{-\mathrm{i}Bt}(B + \mathrm{i})^{-n}(B + \mathrm{i})^{n}\varphi = \chi(B + \mathrm{i})^{-n}e^{-\mathrm{i}Bt}(B + \mathrm{i})^{n}\varphi. \tag{13}$$

Since $(B+\mathrm{i})^n\varphi \in P_{\mathrm{ac}}(B)\mathcal{H}$, the Riemann-Lebesgue lemma implies that $e^{-\mathrm{i}Bt}(B+\mathrm{i})^n\varphi$ converges weakly to 0. Since $\chi(B+\mathrm{i})^{-n}$ is compact, the RHS of (13) converges strongly to 0. □

We now turn to the complex of ideas that comprises the Kato-Birman theory. We start with a definition. Let B be a self-adjoint operator. We denote by $\mathcal{M}(B)$ the set of $\varphi \in \mathcal{H}$ such that $\mathrm{d}\mu_\varphi(\lambda) = |f(\lambda)|^2\mathrm{d}\lambda$ (where $\mathrm{d}\mu_\varphi$ is the spectral measure of φ for B) with $f \in L^\infty(\mathbb{R})$. Let $\|\varphi\|_{\mathcal{M}}$ be the L^∞ norm of f.

It can be verified that $\|\cdot\|_{\mathcal{M}}$ is a norm on $\mathcal{M}(B)$. Furthermore, if $\psi \in P_{\mathrm{ac}}(B)\mathcal{H}$, then $\mathrm{d}\mu_\psi(\lambda) = |f(\lambda)|^2\mathrm{d}\lambda$ for some $f \in L^2(\mathbb{R})$. If $f_n(\lambda) := f(\lambda)\chi_{\{|f|\le n\}}(\lambda)$, then by DCT $f_n \to f$ in $L^2(\mathbb{R})$. Since $\operatorname{supp} f_n \subset \operatorname{supp} f \subseteq \operatorname{sp}_{\mathrm{ac}}(B)$, there is a $\varphi_n \in P_{\mathrm{ac}}(B)\mathcal{H}$ so that $\mathrm{d}\mu_{\varphi_n} = |f_n|^2\mathrm{d}\lambda$. Furthermore φ_n converges to ψ in the usual norm on $\mathcal{H}$, and so $\mathcal{M}(B)$ is dense in $P_{\mathrm{ac}}(B)\mathcal{H}$.

Lemma 1. *For any $\varphi \in \mathcal{M}(B)$ and any $\psi \in \mathcal{H}$,*

$$\int |\langle \psi, e^{-\mathrm{i}tB}\varphi\rangle|^2\mathrm{d}t \le 2\pi\,\|\psi\|^2\,\|\varphi\|_{\mathcal{M}}^2 \tag{14}$$

Proof. Let Q be the projection onto the cyclic subspace generated by B and φ, and let $\mathrm{d}\mu_\varphi = |f|^2\mathrm{d}\lambda$. It is clear that $Q\mathcal{H}$ is unitarily equivalent to $L^2(\mathbb{R}, |f|^2\mathrm{d}\lambda)$, with φ mapping to the function $\varphi(\lambda) \equiv 1$ and $e^{-\mathrm{i}tB}$ acting as multiplication by $e^{-\mathrm{i}t\lambda}$. Let $\eta(\lambda)$ be the function in $L^2(\mathbb{R}, |f|^2\mathrm{d}\lambda)$ corresponding to the vector $Q\psi$. We have,

$$\langle \psi, e^{-\mathrm{i}tB}\varphi\rangle = \langle \psi, Qe^{-\mathrm{i}tB}\varphi\rangle = \langle Q\psi, e^{-\mathrm{i}tB}\varphi\rangle = \int \eta(\lambda)|f(\lambda)|^2 e^{-\mathrm{i}t\lambda}\mathrm{d}\lambda, \tag{15}$$

and so by the Plancherel theorem,

$$\begin{aligned}\int |\langle \psi, e^{-\mathrm{i}tB}\varphi\rangle|^2\mathrm{d}t &= 2\pi \int |\eta(\lambda)|^2|f(\lambda)|^4\mathrm{d}\lambda \\ &\le 2\pi\,\|f\|_\infty \int |\eta(\lambda)|^2|f(\lambda)|^2\mathrm{d}\lambda \\ &= 2\pi\,\|\varphi\|_{\mathcal{M}}^2\,\|Q\psi\|^2 \le 2\pi\,\|\varphi\|_{\mathcal{M}}^2\,\|\psi\|^2\,.\end{aligned}$$

□

The correspondence between the unitary group $e^{\mathrm{i}tB}$ and the Fourier transform also yields

Lemma 2. *For any $\varphi \in P_{\mathrm{ac}}(B)\mathcal{H}$, $e^{-\mathrm{i}tB}\varphi$ converges to 0 weakly as $t \to \pm\infty$. As a consequence, if C is compact, then $Ce^{-\mathrm{i}tB}\varphi$ converges to 0.*

Proof. For any $\psi \in \mathcal{H}$, (15) (which holds even if φ is not in $\mathcal{M}(B)$) gives us that $\langle \psi, e^{-\mathrm{i}tB}\varphi\rangle$ is the Fourier transform of an $L^1(\mathbb{R})$ function (recall that f and ηf are both in $L^2(\mathbb{R})$), and so by the Riemann-Lebesgue lemma, $\langle \psi, e^{-\mathrm{i}tB}\varphi\rangle$ converges to 0 as $t \to \pm\infty$. □

We now prove,

Theorem 7 (Pearson's theorem)**.** *Let A and B be self-adjoint and J a bounded operator. Suppose that there is a trace-class operator C so that $C = AJ - JB$, in the sense that for every $\varphi \in D(A)$ and $\psi \in D(B)$, C satisfies*

$$\langle \varphi, C\psi\rangle = \langle A\varphi, J\psi\rangle - \langle \psi, JB\psi\rangle. \tag{16}$$

Note that since $D(A)$ and $D(B)$ are dense, this defines C uniquely. Then the strong limits

$$\Omega^{\pm}(A, B; J) := \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}At}Je^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)$$

exist.

Remark. Note that (16) implies that $C^* = J^*A - BJ^*$ (in the same sense) and that $JD(B) \subseteq D(A^*) = D(A)$ and $J^*D(A) \subseteq D(B^*) = D(B)$.

Proof. Let $W(t) = e^{\mathrm{i}At}Je^{-\mathrm{i}Bt}$ and consider the case $t \to \infty$. It suffices to prove that

$$\lim_{t<s,t\to\infty} \|(W(t) - W(s))\varphi\|^2 = 0 \tag{17}$$

for every $\varphi \in \mathcal{M}(B)$. Let

$$F_{ab}(X) = \int_a^b e^{\mathrm{i}Bt}Xe^{-\mathrm{i}Bt}\mathrm{d}t. \tag{18}$$

We first show that

$$W(t)^*W(s) - e^{\mathrm{i}aB}W(t)^*W(s)e^{-\mathrm{i}sB} = F_{0a}(Y(t,s)) \tag{19}$$

where,

$$Y(t,s) = -\mathrm{i}[e^{\mathrm{i}tB}J^*e^{-\mathrm{i}(t-s)A}Ce^{-\mathrm{i}sB} - e^{\mathrm{i}tB}C^*e^{-\mathrm{i}(t-s)A}Je^{-\mathrm{i}sB}]. \tag{20}$$

To prove (19), it suffices to consider the matrix elements of both sides, with respect to two vectors in $D(B)$. So, fix φ and ψ in $D(B)$ and let

$$\begin{aligned} Q(b) &= \langle \psi, e^{\mathrm{i}bB}W(t)^*W(s)e^{-\mathrm{i}bB}\varphi\rangle \\ &= \langle W(t)e^{-\mathrm{i}bB}\psi, W(s)e^{-\mathrm{i}bB}\varphi\rangle. \end{aligned}$$

Since $D(B)$ is preserved under the unitary group $e^{\mathrm{i}tB}$, $Q(b)$ is differentiable and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}b}Q(b) \\ &= \mathrm{i}[\langle e^{\mathrm{i}At}JBe^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, e^{\mathrm{i}As}Je^{-\mathrm{i}Bs}e^{-ibB}\varphi\rangle - \langle e^{\mathrm{i}At}Je^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, e^{\mathrm{i}As}JBe^{-\mathrm{i}Bs}e^{-\mathrm{i}bB}\varphi\rangle] \\ &= \mathrm{i}[\langle Be^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, J^*e^{-\mathrm{i}At}e^{\mathrm{i}As}Je^{-\mathrm{i}Bs}e^{-ibB}\varphi\rangle - \langle e^{-\mathrm{i}As}e^{\mathrm{i}At}Je^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, JBe^{-\mathrm{i}Bs}e^{-\mathrm{i}bB}\varphi\rangle] \end{aligned} \tag{21}$$

By the Remark after the theorem, the equalities

$$\begin{aligned} \langle Be^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, J^*e^{-\mathrm{i}At}e^{\mathrm{i}As}Je^{-\mathrm{i}Bs}e^{-ibB}\varphi\rangle &= \langle e^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, J^*Ae^{-\mathrm{i}At}e^{\mathrm{i}As}Je^{-\mathrm{i}Bs}e^{-ibB}\varphi\rangle \\ &\quad - \langle e^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, C^*e^{-\mathrm{i}At}e^{\mathrm{i}As}Je^{-\mathrm{i}Bs}e^{-ibB}\varphi\rangle, \end{aligned}$$

and

$$\begin{aligned} \langle e^{-\mathrm{i}As}e^{\mathrm{i}At}Je^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, JBe^{-\mathrm{i}Bs}e^{-\mathrm{i}bB}\varphi\rangle &= \langle Ae^{-\mathrm{i}As}e^{\mathrm{i}At}Je^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, Je^{-\mathrm{i}Bs}e^{-\mathrm{i}bB}\varphi\rangle \\ &\quad - \langle e^{-\mathrm{i}As}e^{\mathrm{i}At}Je^{-\mathrm{i}Bt}e^{-\mathrm{i}bB}\varphi, Ce^{-\mathrm{i}Bs}e^{-\mathrm{i}bB}\varphi\rangle \end{aligned}$$

are justified. Substituting these two equalities into (21) yields,

$$\frac{\mathrm{d}}{\mathrm{d}b}Q(b) = -\langle \psi, e^{\mathrm{i}bB}Y(t,s)e^{-\mathrm{i}bB}\varphi\rangle.$$

Integrating from 0 to a proves (19).

For fixed t and s and $\psi \in D(A)$, $\varphi \in D(B)$, the formula

$$\langle \psi, (W(t) - W(s))\psi\rangle = \mathrm{i}\int_s^t \langle \psi, e^{\mathrm{i}uA}Ce^{-\mathrm{i}uB}\varphi\rangle \mathrm{d}u$$

holds, and so by density of $D(B)$ in $\mathcal{H}$ it follows that $W(t) - W(s)$ is compact. By Lemma 2,

$$\lim_{a\to\infty} e^{\mathrm{i}aB}W(t)^*(W(t) - W(s))e^{-\mathrm{i}aB}\varphi = 0, \tag{22}$$

for any $\varphi \in \mathcal{M}(B)$. By (19). It then follows immediately that for any $\varphi \in \mathcal{M}(B)$,

$$\langle \varphi, W(t)^*(W(t) - W(s))\varphi\rangle = \lim_{a\to\infty}\langle \varphi, F_{0a}(Y(t,t) - Y(t,s))\varphi\rangle. \tag{23}$$

Since C is trace class, we can write

$$C = \sum_{n=1}^{\infty} \lambda_n \langle \varphi_n, \cdot \rangle \psi_n$$

where the φ_n's and the ψ_n's are orthonormal, the λ_n's are all strictly positive and $\sum \lambda_n = \|C\|_1 < \infty$, where $\|\cdot\|_1$ is the trace norm. We apply this to derive, for any bounded operator X and $a > 0$,

$$\begin{aligned}
|\langle \varphi, F_{0a}(e^{\mathrm{i}uB} X C e^{-\mathrm{i}uB})\varphi\rangle| &\leq \sum_n \lambda_n \int_u^{a+u} |\langle e^{-\mathrm{i}xB}\varphi, X\psi_n\rangle\langle\psi_n, e^{-\mathrm{i}xB}\varphi\rangle| \mathrm{d}x \\
&\leq \sum_n \lambda_n \left[\int_u^{u+a} |\langle e^{-ixB}\varphi, X\psi_n\rangle|^2 \mathrm{d}x\right]^{1/2} \left[\int_u^{u+a} |\langle\psi_n, e^{-\mathrm{i}xB}\varphi\rangle|^2 \mathrm{d}x\right]^{1/2} \\
&\leq \left[\sum_n \lambda_n \int_{-\infty}^{\infty} |\langle X\psi_n, e^{-\mathrm{i}xB}\varphi\rangle|^2 \mathrm{d}x\right]^{1/2} \left[\sum_n \lambda_n \int_u^{\infty} |\langle\psi_n, e^{-\mathrm{i}xB}\varphi\rangle|^2 \mathrm{d}x\right]^{1/2} \\
&\leq (2\pi \|C\|_1)^{1/2} \|X\| \|\psi\|_{\mathcal{M}} \left[\sum_n \lambda_n \int_u^{\infty} |\langle\psi_n, e^{-\mathrm{i}xB}\varphi\rangle|^2 \mathrm{d}x\right]^{1/2}
\end{aligned}$$

The first inequality follows from our expansion for C. The second and third inequalities are Cauchy-Schwartz, and the last is Lemma 1. Clearly the same inequality holds if XC is substituted with C^*X. This, together with (23) yield,

$$\|(W(t) - W(s))\varphi\|^2 \leq 8(2\pi \|C\|_1)^{1/2} \|\varphi\|_{\mathcal{M}} \|J\| \left[\sum_n \lambda_n \int_{\min\{t,s\}}^{\infty} |\langle\psi_n, e^{-\mathrm{i}xB}\varphi\rangle|^2 \mathrm{d}x\right]^{1/2}. \quad (24)$$

One corollary of the above inequality, which we record here for later use, is (using Lemma 1),

$$\|(W(t) - W(s))\varphi\|^2 \leq 16\pi \|C\|_1 \|\varphi\|_{\mathcal{M}}^2 \|J\|. \quad (25)$$

Finally, Lemma 1 gives us that the function $x \to \sum_n \lambda_n |\langle\psi_n, e^{-\mathrm{i}xB}\varphi\rangle|^2$ is in $L^1(\mathbb{R})$, and so (17), and therefore the theorem, follows. □

Taking $s = 0$ and $t \to \pm\infty$ in (27) yields

Corollary 1. *Under the hypotheses of Theorem 7,*

$$\left\|\Omega^{\pm}(A, B; J) - J\right\|^2 \leq 16\pi \|C\|_1 \|\varphi\|_{\mathcal{M}}^2 \|J\|, \quad (26)$$

for $\varphi \in \mathcal{M}(B)$.

Note that $C = AJ - JB$ also implies $C^* = J^*A - BJ^*$ (in the sense of Theorem 7), and so both s-lim $e^{\mathrm{i}At} J e^{-\mathrm{i}Bt} P_{\mathrm{ac}}(B)$ and s-lim $e^{\mathrm{i}tB} J^* e^{-\mathrm{i}tA} P_{\mathrm{ac}}(A)$ exist. For general J, this does not imply completeness of either strong limit. However, if $J = 1$, then we can apply Proposition 6 to conclude

Theorem 8 (Kato-Rosenblum theorem)**.** *If A and B are self-adjoint and $A - B$ is trace-class (in the sense of Theorem 7) then $\Omega^{\pm}(A, B)$ exist and are complete.*

Remark. It follows that $D(A) = D(B)$.

We also have,

Proposition 7. *Let $\{A_n\}_{n=1}^{\infty}$, A and B be self-adjoint operators.Suppose that the wave operators $\Omega^{\pm}(A, B)$exist and that each $A_n - A$ is trace class in the sense of Theorem 7 with $\|A_n - A\|_1 \to 0$ as $n \to \infty$. Then, for every n, the wave operators $\Omega^{\pm}(A_n, B)$ exist and*

$$\Omega^{\pm}(A, B) = \operatorname*{s-lim}_{n\to\infty} \Omega^{\pm}(A_n, B).$$

If in addition, $\Omega^{\pm}(B,A)$ exist then for every n the wave operators $\Omega^{\pm}(B,A_n)$ exist and

$$\Omega^{\pm}(B,A)\varphi = \lim_{n\to\infty} \Omega^{\pm}(B,A_n)\varphi$$

for every $\varphi \in \operatorname{ran} P_{ac}(A)$.

Proof. By Proposition 5 and Theorem 8, $\Omega^{\pm}(A_n,B) = \Omega^{\pm}(A_n,A)\Omega^{\pm}(A,B)$ for every n. To prove the first claim it suffices to prove

$$\operatorname*{s-lim}_{n\to\infty} \Omega^{\pm}(A_n,A) = P_{\mathrm{ac}}(A). \tag{27}$$

This follows immediately from Corollary 1, for if $\varphi \in \mathcal{M}(A)$,

$$\left\|(\Omega^{\pm}(A_n,A) - P_{\mathrm{ac}})\varphi\right\| = \left\|(\Omega^{\pm}(A_n,A) - \mathbb{1})\varphi\right\| \le 16\pi \left\|A_n - A\right\|_1 \left\|\varphi\right\|_{\mathcal{M}}^2,$$

which goes to 0 as $n \to \infty$. Since $\Omega^{\pm}(A_n,A)$ is 0 on $(P_{\mathrm{ac}}(A)\mathcal{H})^{\perp}$ and $\mathcal{M}(A)$ is dense in $P_{\mathrm{ac}}(A)\mathcal{H}$, we conclude (27).

Again by Proposition 5 and Theorem 8, $\Omega^{\pm}(B,A_n) = \Omega^{\pm}(B,A)\Omega^{\pm}(A,A_n)$ so to prove the second claim it suffices to prove

$$\lim_{n\to\infty} \Omega^{\pm}(A,A_n)\varphi = \varphi \tag{28}$$

for any $\varphi \in \operatorname{ran} P_{\mathrm{ac}}(A)$. Let $\varphi_n = \Omega^{\pm}(A_n,A)\varphi$. By (27), $\|\varphi_n - \varphi\| \to 0$, and so

$$\lim_{n\to\infty} \left\|\Omega^{\pm}(A,A_n)(\varphi_n - \varphi)\right\| = 0.$$

Since $\Omega^{\pm}(A,A_n)\varphi_n = \Omega^{\pm}(A,A_n)\Omega^{\pm}(A_n,A)\varphi = P_{\mathrm{ac}}(A)\varphi = \varphi$, this yields (31). ☐

While Theorem 8 is quite general it is not always useful. In quantum mechanics, $A - B$ may not even be bounded. We require some generalizations.

Theorem 9 (Kuroda-Birman theorem)**.** *Let A and B be self-adjoint operators with $(A+\mathrm{i})^{-1} - (B+\mathrm{i})^{-1}$ trace-class. Then the wave operators $\Omega^{\pm}(A,B)$ exist and are complete.*

Proof. Let $J = (A+\mathrm{i})^{-1}(B+\mathrm{i})^{-1}$. For $\psi \in D(A)$ and $\varphi \in D(B)$,

$$\langle A\psi, J\varphi\rangle - \langle \psi, JB\varphi\rangle = \langle \psi, ((B+\mathrm{i})^{-1} - (A+\mathrm{i})^{-1})\varphi\rangle, \tag{29}$$

i.e., $AJ - JB = (B+\mathrm{i})^{-1} - (A+\mathrm{i})^{-1}$ in the sense of Theorem 7 and since the RHS is trace-class by assumption, we conclude that

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{\mathrm{i}At}(A+\mathrm{i})^{-1}(B+\mathrm{i})^{-1}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)$$

exist. Any vector $\varphi \in D(B)$ we have,

$$e^{\mathrm{i}At}(A+\mathrm{i})^{-1}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}\varphi = e^{\mathrm{i}At}(A+\mathrm{i})^{-1}(B+\mathrm{i})^{-1}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)(B+\mathrm{i})\varphi$$

and so we conclude that the limits $t \to \pm\infty$ on the LHS exist for every $\varphi \in D(B)$, and so by density we conclude that the strong limits

$$\operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}At}(A+\mathrm{i})^{-1}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B).$$

exist. Since $(A+\mathrm{i})^{-1} - (B+\mathrm{i})^{-1}$ is compact, we by Lemma 2,

$$\operatorname*{s-lim}_{t\to\pm\infty} ((A+\mathrm{i})^{-1} - (B+\mathrm{i})^{-1})e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B) = 0.$$

It follows that

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{\mathrm{i}At}(B+\mathrm{i})^{-1}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)$$

exists. Now,

$$e^{\mathrm{i}At}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)\varphi = e^{\mathrm{i}At}(B+\mathrm{i})^{-1}e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)(B+\mathrm{i})\varphi \tag{30}$$

for $\varphi \in D(B)$, and so we conclude that $\Omega^{\pm}(A,B)$ exist. By symmetry, $\Omega^{\pm}(B,A)$ exist and the proof is complete. □

We require the following definition. Let A and B be self-adjoint. We say that A is **subordinate** to B if there are continuous functions f and g on $\mathbb{R}$ with $f(x) \geq 1$ and $g(x) \geq 1$ and

$$\lim_{|x|\to\infty} f(x) = \infty$$

so that $D(g(B)) \subseteq D(f(A))$ with $f(A)g(B)^{-1}$ bounded. If A is subordinate to B and B is subordinate to A, we say that A and B are **mutually subordinate**.

This condition is very weak. For example, if $D(A) = D(B)$, then A and B are mutually subordinate. For this, consider $f(x) = g(x) = 1 + |x|$. Then $g(B)^{-1}\mathcal{H} \subseteq D(A)$ and so by the closed graph theorem, $f(A)g(B)^{-1}$ is bounded, and vice versa.

Theorem 10 (Birman's theorem)**.** *Suppose that A and B are self adjoint with spectral projections $E_\Omega(A)$ and $E_\Omega(B)$ respectively, for Ω Borel. Assume that*

(i) *$E_I(A)(A-B)E_I(B)$ is trace-class for every bounded interval I.*

(ii) *A and B are mutually subordinate.*

Then the wave operators $\Omega^{\pm}(A,B)$ exist and are complete.

Proof. By the symmetry on the hypotheses and Proposition 6, it suffices to show that $\Omega^{\pm}(A,B)$ exist. Let $E_a(C) := E_{(-a,a)}(C)$ and $E'_a(C) := E_{(-\infty,a]\cup[a,\infty)}(C)$ where C is A or B. If $J = E_a(A)E_b(B)$, then J is trace-class by hypothesis. Therefore,

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{\mathrm{i}At}E_a(A)E_b(B)e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)$$

exist by Theorem 7. The set

$$\bigcup_{a>0} \operatorname{ran} E_a(B)$$

is dense in $\mathcal{H}$. Let φ be an element of this set, i.e., $\varphi \in \operatorname{ran} E_{a_0}(B)$ for some $a_0 > 0$. For $a > a_0$ we have that

$$\lim_{t\to\pm\infty} e^{\mathrm{i}At}E_a(A)e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)\varphi$$

exists. Therefore, to conlude the theorem we need only show

$$\lim_{a\to\infty}\left[\sup_t \left\|E'_a(A)e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)\varphi\right\|\right] = 0. \tag{31}$$

Let f and g be the functions guaranteed by the condition that A is subordinate to B. Let $F(a) = \inf_{|x|\geq a} f(x)$. Then $F(a) \to \infty$ as $a \to \infty$. Therefore,

$$\begin{aligned}\left\|E'_a(A)e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)\varphi\right\| &\leq F(a)^{-1}\left\|f(A)E'_a(A)e^{-\mathrm{i}Bt}P_{\mathrm{ac}}(B)\varphi\right\| \\ &\leq F(a)^{-1}\left\|f(A)E'_a(A)g(B)^{-1}\right\|\left\|g(B)e^{-\mathrm{i}Bt}\varphi\right\| \\ &\leq F(a)^{-1}\left\|f(A)g(B)^{-1}\right\|\left[\sup_{|x|\leq a_0}|g(x)|\right]\|\varphi\|\end{aligned}$$

from which we conclude (31). In the last inequality we have used $f(A)E'_a(A) = E'_a(A)f(A)$. □

Numerous other conditions on A and B arise in applications that have not been covered by the above results. For example if A and B are positive operators and $A^2 - B^2$ is trace-class, is it true that $\Omega^{\pm}(A,B)$ exist? Or for $A = -\Delta + V$ and $B = -\Delta$ (on $\mathbb{R}^n$), $(A+\mathrm{i})^{-1} - (B+\mathrm{i})^{-1}$ is not trace-class for any nontrivial potential V for $n \geq 4$. But for large E and k, $(A+E)^{-k} - (B+E)^{-k}$ for certain potentials V. Does this imply existence of the wave operators?

We turn now to an abstract theorem which will allow us to address these questions. We first require the following notion: a function φ on T, with T an open subset of $\mathbb{R}$ is said to be **admissable** if $T = \bigcup_{n=1}^{N} I_n$ where the I_n are finite disjoint open intervals, N infinite or finite and

(i) The distributional derivative φ'' is an L^1 function on every compact subinterval of T ,
(ii) On each I_n, φ' is either strictly positive or strictly negative.

As a consequence of (i), φ is C^1 on compact subintervals of each I_n.
Example. Let $T = (0, \infty)$. The function $\varphi(x) = x^{1/2}$ is admissible. Note that if $A^2 = A_1$ and $B^2 = B^1$, then as long as A and B are positive, then $A = \varphi(A_1)$ and $B = \varphi(B_1)$, and $A_1 - B_1$ is trace-class if $A^2 - B^2$ is trace-class.
Example. If $T = (0, \infty)$, then $\varphi(x) = x^{-1/n} - a$ is admissible. Let A and B be self-adjoint satisfying $A > -a$ and $B > -a$. Let $A_1 = (A + a)^{-n}$ and $B_1 = (B + a)^{-n}$. Then $A = \varphi(A_1)$, $B = \varphi(B_1)$ and $A_1 - B_1$ is trace-class if $(A + a)^{-n} - (B + a)^{-n}$ is trace-class.

The notion of admissibility is useful as the following theorem shows:

Theorem 11. *Let φ be an admissible function an open set T. Suppose that A and B are self-adjoint operators with $\sigma(A), \sigma(B) \subseteq \bar{T}$ and that at each boundary point of T either φ has a finite limit, or both A and B do not have point spectrum at that point. Suppose that $A - B$ is trace-class. Then $\Omega^{\pm}(\varphi(A), \varphi(B))$ exist, are complete and*

$$\Omega^{\pm}(\varphi(A), \varphi(B)) = \Omega^{\pm}(A, B)E_{T_1}(B) + \Omega^{\mp}(A, B)E_{T_2}(B)$$

where T_1 (resp., T_2) is the union of those intervals where $\varphi' > 0$ (resp., $\varphi' < 0$).

As a consequence of this theorem and the two examples above, we have immediately the following corollaries.

Corollary 2. *If A and B are positive operators with $A^2 - B^2$ trace-class, then $\Omega^{\pm}(A, B)$ exist and are complete.*

Corollary 3. *If A and B are positive operators with $(A^2 + 1)^{-1} - (B^2 + 1)^{-1}$ trace-class then $\Omega^{\pm}(A, B)$ exist and are complete.*

Corollary 4. *If A and B are self-adjoint operators with $A, B \geq -a + 1$ and $(A + a)^{-k} - (B + a)^{-k}$ trace-class, then $\Omega^{\pm}(A, B)$ exist and are complete.*

We require the following lemma for the proof of Theorem 11.

Lemma 3. *Let φ be an admissible function. Then,*

(i) *If $Y \subset \mathbb{R}$ has Lebesgue measure 0 than so do $\varphi[Y \cap T]$ and $\varphi^{-1}[Y]$.*
(ii) *For any $w \in L^2(I_n)$ with $\varphi' > 0$ on I_n,*

$$\lim_{s \to \infty} \int_0^{\infty} \left| \int_{-} \infty^{\infty} e^{\mathrm{i}(t\lambda + s\varphi(\lambda))} w(\lambda) \mathrm{d}\lambda \right|^2 \mathrm{d}t = 0. \tag{32}$$

If $\varphi' < 0$ on I_n, then the limit $s \to \infty$ is replaced with $s \to -\infty$.

Proof. (i) is an easy consequence of the fact that φ is strictly monotone on each I_n. We will prove (ii) in the case that $\phi; > 0$ on I_n. The other case is similar. Since

$$(2\pi)^{-1/2} \int_{\mathbb{R}} e^{\mathrm{i}(t\lambda + s\varphi(\lambda))} w(\lambda) \mathrm{d}\lambda$$

is the inverse Fourier transform of $e^{\mathrm{i}s\varphi(\lambda)} w(\lambda)$ we have

$$2\pi \|w\|^2 \geq \int_0^{\infty} \left| \int_{-\infty}^{\infty} e^{\mathrm{i}(t\lambda + s\varphi(\lambda))} w(\lambda) \mathrm{d}\lambda \right|^2 \mathrm{d}t.$$

It follows that we need only prove (32) for w the characteristic function of some $[a,b] \subseteq I_n$. Since φ is C^1 on $[a,b]$ we have $\inf_{[a,b]} \varphi' = \gamma > 0$. We have for $t > 0$ and $s > 0$,

$$e^{-\mathrm{i}(t\lambda + s\varphi(\lambda))} = \mathrm{i}(t + s\varphi'(\lambda))^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda}(e^{-\mathrm{i}(t\lambda + s\varphi(\lambda))}).$$

And so,

$$\begin{aligned}\left| \int_a^b e^{-\mathrm{i}(t+s\varphi(\lambda))} \mathrm{d}\lambda \right| &= \left| \int_a^b (t + s\varphi'(\lambda))^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda} e^{-\mathrm{i}(t\lambda + s\varphi(\lambda))} \mathrm{d}\lambda \right| \\ &\leq (t + s\varphi'(b))^{-1} + (t + s\varphi'(a))^{-1} + (t + s\gamma)^{-2} s \int_a^b |\varphi''(\lambda)|^2 \mathrm{d}\lambda.\end{aligned}$$

Above, we have used integration by parts. This function is in $L^2(0,\infty)$ and goes to 0 as $s \to \infty$ by DCT. □

The proof of Lemma 3 as gives us,

Corollary 5. *(32) also holds if the* $-\mathrm{i}$ *in the exponential is changed to* $+\mathrm{i}$ *and the limit* $s \to \pm\infty$ *is changed to* $s \to \mp\infty$.

Proof of Theorem 11. Let $C = A - B = \sum \lambda_n \langle \psi_n, \cdot \rangle \psi_n$ and let

$$\eta \in \operatorname{ran} E_{I_n}(B) \cap \mathcal{M}(B).$$

The proof of Theorem 7, specifically taking $s = 0$ in (24), yields

$$\left\| (\Omega^-(A,B) - \mathbb{1}) e^{-\mathrm{i}\varphi(B)s} \eta \right\|^2 \leq c \left[\sum_n |\lambda_n| \int_0^\infty \langle \psi_n, e^{-\mathrm{i}Bt - \mathrm{i}\varphi(B)s} \eta \rangle|^2 \mathrm{d}t \right]^{1/2} \tag{33}$$

for some constant c (note that c includes the norm $\|e^{-\mathrm{i}s\varphi(B)}\eta\|_{\mathcal{M}} = \|\eta\|_{\mathcal{M}}$ which does not depend on s since φ is real valued.) Lemma 3(2) implies that each of the integrals on the RHS of (34) goes to 0 as $s \to \infty$ (resp., $s \to -\infty$) if $\varphi' > 0$ (resp., $\varphi' < 0$) on I_n. Lemma 1 implies that each integral is bounded by $2\pi \|\psi_n\|^2 \|\eta\|_{\mathcal{M}}$ for every s. Since $\sum |\lambda_n| \|\psi_n\|^2 = \|C\|_1 < \infty$, we have, by DCT for sums, that the RHS of (34) goes to 0 for $s \to \pm\infty$. Proposition 4 tells us that $A\Omega^\pm(A,B) = \Omega^\pm(A,B)B$ and so it is a consequence of the Borel functional calculus that $e^{-\mathrm{i}\varphi(A)s}\Omega^\pm(A,B) = \Omega^\pm(A,B)e^{-i\varphi(B)s}$. We have therefore derived,

$$\lim_{s\to\infty} e^{\mathrm{i}\varphi(A)s} e^{-\mathrm{i}\varphi(B)s} \eta = \Omega^-(A,B)\eta$$

if $\varphi' > 0$, and then the same thing with the limit $s \to \infty$ replaced by $s \to -\infty$ if $\varphi' < 0$. If we repeat the same proof as above, but replace the usage of the inequality (24) with the corresponding inequality for $\Omega^+(A,B)$ and the use of Lemma 3(ii) with Corollary 5, we obtain

$$\lim_{s\to\infty} e^{\mathrm{i}\varphi(A)s} e^{-\mathrm{i}\varphi(B)s} \eta = \Omega^+(A,B)\eta$$

for $\varphi' < 0$, and then the same thing with the limit replaced by $s \to -\infty$ if $\varphi' > 0$. Lemma 3(i) and the additional hypotheses on $\mathrm{sp}(B)$ and φ (i.e., with regards to the boundary points of T) guarantee that $P_{\mathrm{ac}}(\varphi(B)) = P_{\mathrm{ac}}(B)$ and so we conclude that the wave operators $\Omega^\pm(\varphi(A), \varphi(B))$ exist. By symmetry, $\Omega^\pm(\varphi(B), \varphi(A))$ exist and we conclude the theorem. □

Remark. The hypothesis that $A - B$ is trace-class can be replaced by either the hypotheses of Birman's theorem or those of the Kuroda-Birman theorem. The proofs proceed by proving the analog of Pearson's theorem under this general set-up; that if $AJ - JB$ is trace-class, then the strong limits

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{\mathrm{i}\varphi(A)t} J e^{-\mathrm{i}\varphi(B)t} P_{\mathrm{ac}}(\varphi(B))$$

exist. One then uses the same J's as used in the proofs of those two theorems.

Let us conclude with the following result about the scattering theory of Schrödinger operators.

Theorem 12 (Cook-Hack theorem). *Let $V \in L^2(\mathbb{R}^3) + L^r(\mathbb{R}^3)$ where $r < 3$. Let $H_0 = -\Delta$ and $H = -\Delta + V$. Then $\Omega^{\pm}(H, H_0)$ exist.*

Proof. Fix $\gamma > 0$ and let

$$\varphi_\gamma(x) = \gamma^{3/4} e^{-\frac{1}{2}\gamma x^2}.$$

The Fourier transform of φ_γ is $\hat{\varphi}_\gamma(p) = Ce^{-\frac{1}{2}p^2/\gamma}$ for some constant C, so $(e^{-\mathrm{i}tH_0}\varphi_\gamma)^\wedge = Ce^{-\frac{1}{2}p^2/\gamma(t)}$ where $\gamma(t)^{-1} = \gamma^{-1} - 2\mathrm{i}t$. The constant can be evaluted using $\|e^{-\mathrm{i}tH_0}\varphi_\gamma\| = \|\varphi_\gamma\|$. One obtains

$$(e^{-\mathrm{i}tH_0}\varphi_\gamma)(x) = \alpha(t)^{3/4} e^{-\frac{1}{2}[\alpha(t)+\mathrm{i}\beta(t)]x^2} \tag{34}$$

where $\alpha(t) = \gamma(1 + 4t^2\gamma^2)^{-1}$ for some suitable real-valued $\beta(t)$. From (34) it is easy to see that, for $k > 0$

$$\left\|(1 + |x|)^k e^{-\mathrm{i}tH_0}\varphi_\gamma\right\|_\infty \le c_\gamma(1 + |t|)^{-3/2+k}. \tag{35}$$

Indeed, differentiating the function, for $r > 0$,

$$(1 + r)^k e^{-\frac{1}{2}\alpha(t)r^2}$$

one finds that the supremum of the function on the LHS of (35) is attained when $|x|$ satisfies

$$\frac{k}{|x|(1 + |x|)} = \frac{\gamma}{1 + 4t^2\gamma^2}.$$

For large t, $|x| \sim t$, and so (35) follows.

We conclude from (35) that

$$\left\|Ve^{-\mathrm{i}tH_0}\varphi_\gamma\right\|_2 \le c\left\|(1 + |x|)^{-k}V\right\|_2 (1 + |t|)^{-3/2+k} \le c'(\|V_2\|_2 + \|V_r\|_r)(1 + |t|)^{-3/2+k}$$

where $V = V_2 + V_r \in L^2 + L^r$, and $r^{-1} = \frac{1}{2} + k/(3 + \varepsilon)$ for some $\varepsilon > 0$. This is a result of Hölder's inequality and the fact that $(1 + |x|)^{-k}$ is in L^m for $m > 3k^{-1}$. Since $r < 3$, we can take $k < 1/2$ and conclude that

$$\int \left\|Ve^{-\mathrm{i}tH_0}\varphi_\gamma\right\|_2 \mathrm{d}t < \gamma.$$

Since linear combinations of translations of φ_γ's with $\gamma > 0$ are dense in $L^2(\mathbb{R}^3)$ and $H_0 = -\Delta$ has purely absolutely continuous spectra, we conclude from Theorem 5 that $\Omega^{\pm}(H, H_0)$ exist. □

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