Introduction to a Probabilistic Approach to PDEs

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Abstract

This final project of PDE 2 taught by Dr. Gantumur Tsotgerel investigates probabilistic methods to solve toy model pde's such as the Dirichlet Problem and the Heat Equation. In the last part, we discuss the Feynman-Kac Representation Theorem which establishes a strong connection between a family of Stochastic Differential Equations and Classical Partial Differential Equations.

1 Introduction

1.1 One Dimensional Brownian Motion

We begin by defining Brownian Motion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space together with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, that is each $\mathcal{F}_t$ is a σ-algebra on Ω and $t \leq s$ implies $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$. A stochastic process $\mathbf{B} = \{B_t\}_{t\geq 0} : [0,\infty) \times \Omega \ni (t,\omega) \to B_t(\omega) \in \mathbb{R}$ is called a Brownian Motion started at x if:

1. it is initially at the point $x \in \mathbb{R}$, that is, $B_0(\omega) = x$ a.s.
2. the sample paths are (a.s.) continuous, that is, for almost all ω in Ω the maps $[0,\infty) \ni t \to B_t(\omega)$ are continuous.
3. the increments of the motion are independent, that is, for all $0 \leq t \leq s < \infty$, the increment $B_s - B_t$ is independent of $\mathcal{F}_t$.
4. the increments are normally distributed, that is, for all $0 \leq t \leq s < \infty$, the distribution of the increment $B_s - B_t$ is that of a Gaussian with mean 0 and variance $s - t :\ N(0, s-t)$

We have a few remarks:

1. The continuous parameter t should be thought of as time. At each moment in time t, the process is represented by the random variable $B_t : \Omega \ni \omega \to B_t(\omega)$.

2. The natural filtration $\{\mathcal{F}_t\}$ chosen is that which makes the the collection of random variables $\{B_t\}_{0\leq t\leq s}$ measurable with respect to $\mathcal{F}_s$, for every $s \in \mathbb{R}$. Thus $\mathcal{F}_s = \sigma(B_t : 0 \leq t \leq s)$. One should think of the σ-algebra $\mathcal{F}_s$ as the collection of events that can or have be measured up until time s. As time grows and more information is collected from observing the process, the number of events increases.

3. Another σ-algebra which is sometimes more useful to use is $\mathcal{F}_t^+ = \bigcap\limits_{s>t} \mathcal{F}_s$, where $\mathcal{F}_s$ is as defined in the previous item. It is obvious that $\{\mathcal{F}_t^+\}_{t\geq 0}$ is also a filtration and that $\mathcal{F}_t^+ \supset \mathcal{F}_t$ for all t, suggesting that this revamped σ-algebra allows an additional infinitesimal glance into the future.

4. $B_s - B_t$ is the difference of two random variables so is a random variable in its own right and measures the displacement of the process between times t and s. $B_s - B_t$ is independent of $\mathcal{F}_t$ means that for all $A \subset \mathbb{R}$ and $B \in \mathcal{F}_t$, $\mathbb{P}((B_s - B_t)^{-1}(A) \cap B) = \mathbb{P}((B_s - B_t)^{-1}A) \cdot \mathbb{P}(B)$.

5. Item (4) in the definition of Brownian Motion means that

   $\mathbb{P}(\{B_s - B_t \in [x_1, x_2]\}) = \frac{1}{\sqrt{2\pi(s-t)}} \int_{x_1}^{x_2} e^{-\frac{\xi^2}{2(s-t)}} d\xi$, where by

   $\{B_s - B_t \in [x_1, x_2]\}$ we mean $\{\omega \in \Omega : B_s(\omega) - B_t(\omega) \in [x_1, x_2]\}$.

1.2 Multidimensional Brownian Motion

By a d-dimensional Brownian Motion started at $x = (x_1, ..., x_d)$ we simply mean a process of the form $\mathbf{B} = \{B_t\}_{t\geq 0} : [0, \infty) \times \Omega \ni (t, \omega) \to (B_t^1(\omega), ..., B_t^d(\omega)) \in \mathbb{R}^d$, where $B^1, ..., B^d$ are independent one dimensional Brownian motions started at $x_1, ..., x_d$.

1.3 The Strong Markov Property

Brownian Motion satisfies the important Strong Markov Property which is a much larger generalization of property (3) in the definition of Brownian Motion. It states that for any finite stopping time T, the process $\{B_{T+t} - B_T\}_{t\geq 0}$ is a Brownian Motion started at 0 independent of $\mathcal{F}_T^+$. At this point, we should also define what a stopping time is and what $\mathcal{F}_T^+$ is, but for the purposes of this report, we shall not go into the full details of probability. It is just important to know that a stopping time T is a random variable, thus a map $\Omega \ni \omega \to T(\omega) \in \mathbb{R}$. At time $T(\omega)$, a flag signals that the process has reached a certain threshhold and the process is stopped. It is important to keep in mind that, in the same way that we are not interested in the individual sample paths of the Brownian Motion, but rather the global behavior of the sample paths, we are not interested in the individual values of $T(\omega)$ for each ω, but rather the random variable T as a whole. Finally $\mathcal{F}_T^+$ can be roughly thought of as the collection of all possible events that could happen before the process is stopped (according to some logic). Of course, deterministic times are stopping times, that is, for any fixed $t \in [0, \infty)$, $T(\omega) \equiv t$ is a well defined stopping time; "just stop the process after t seconds".

One can show that the Strong Markov Property of Brownian Motion described above implies (and is in fact equivalent) to the following statement :

For any bounded function $f : \mathbb{R}^d \to \mathbb{R}$ and for all $x \in \mathbb{R}^d$,

$$\mathbb{E}_x[f(B_{T+t})|\mathcal{F}_t^+] = \mathbb{E}_{B_T}[f(\tilde{B}_t)]$$

where $\tilde{\mathbf{B}}$ is a Brownian Motion seeded at B_T, i.e. $\tilde{B}_0 = B_T$. The subscripts x and B_T on the two expectations are simply there to remind that $\mathbf{B}$ was seeded at x and $\tilde{\mathbf{B}}$ was seeded at B_T. Again we shall not define conditional expectation here, but intuitively this is saying that if the Brownian process is stopped at some random time T, then it is only worthwhile to know the state of the process at time T in order to best guess the future state of the Brownian motion t seconds later, i.e. all history of the process can be thrown by the window except for the current state.

2 The Dirichlet Problem

Let $U \subset \mathbb{R}^d$ be open and connected and $\varphi : \partial U \to \mathbb{R}$ be a continuous function. The Dirichlet problem is to find a continuous solution to the problem

$$\Delta u = 0 \text{ in } U,\ u = \varphi \text{ on } \partial U$$

We recall two standard tools from introductory pde's:

Proposition (Mean Value Property) Let $U \subset \mathbb{R}^d$ be open and connected and $u : U \to \mathbb{R}$ be locally bounded. Then TFAE:

1. u is harmonic on U
2. for any ball $B(x,r) \subset U$, $u(x) = \frac{1}{Leb(B(x,r))} \int_{B(x,r)} u(s)ds$
3. for any ball $B(x,r) \subset U$, $u(x) = \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} u(s)d\sigma$, where σ is the Lebesgue surface measure

Proposition (Maximum Principle) Let $u : \mathbb{R}^d \to \mathbb{R}$ be subharmonic on the open connected set U. Then

1. u does not achieve a maximum on U unless it is constant.
2. if further u is continous on $\overline{U}$ and U is bounded, then $\max\limits_{x \in \overline{U}} u(x) = \max\limits_{x \in \partial U} u(x)$

Proposition Let U be open, connected and bounded, and $\mathbf{B} = \{B_t\}_{t \geq 0}$ be a Brownian motion started inside U. Define the stopping time $\tau(\partial U) = \tau =$

$\min\{t \geq 0 : B_t \in \partial U\}$ which is finite since U is bounded. If $\varphi : \partial U \to \mathbb{R}$ is such that the function $u : U \to \mathbb{R}$, $u(x) = \mathbb{E}_x[\varphi(B_\tau)]$ is locally bounded, then u is harmonic on U.

Notation $\tau : \Omega \to \tau(\omega) = \min\{t \geq 0 : B_t(\omega) \in \partial U\} \in \mathbb{R}$ is the first time the Brownian motion hits the boundary of U.

Proof

Since we are assuming local boundedness, by the mean value property, it suffices to check that for all balls $B(x,r) \subset U$, $u(x) = \int_{\partial B(x,r)} u d\alpha$ where $d\alpha$ denotes the normalized Lebesgue surface measure of $B(x,r)$.

Let $B(x,r) \subset U$, $\mathbf{B}$ be a Brownian motion started at x and $\tilde{\tau} = \min\{t \geq 0 : B_t \notin B(x,r)\}$. So $\tilde{\tau}$ is the first time the Brownian motion exits the ball $B(x,r)$. We must then have $\tau \geq \tilde{\tau}$ and so $\varphi(B_\tau)$ is measurable with respect to $\mathcal{F}^+_{\tilde{\tau}}$ so by the properties of conditional expectation

$$u(x) = \mathbb{E}_x[\mathbb{E}_x[\varphi(B_\tau)|\mathcal{F}^+_{\tilde{\tau}}]]$$

By the Strong Markov property,

$\mathbb{E}_x[\varphi(B_\tau)|\mathcal{F}^+_{\tilde{\tau}}] = \mathbb{E}_{B(\tilde{\tau})}[\varphi(B_\tau)] = u(B(\tilde{\tau}))$. Hence $u(x) = \mathbb{E}_x[u(B(\tilde{\tau}))]$. Since $\mathbf{B}$ starts at the center of the ball $B(x,r)$ and propagates in all directions with the same probability (gaussian), $u(x) = \mathbb{E}_x[u(B(\tilde{\tau}))] = \int_{\partial B(x,r)} u d\alpha$ as required.

In order to continue the probabilistic approach, we need to impose some regularity conditions on the domain U. We say that U satisfies the Poincare cone condition at $x \in \partial U$ if there exists a cone C_x (with non zero opening angle) based at x and $r > 0$ such that $C_x \cap B(x,r) \subset U^c$. Upon investigation of the proofs to come, we could relax the condition to hold for cone-like shapes, with sharper tips for example.

Lemma Let $C_0 \subset \mathbb{R}^d$ be a cone (like shape) based at the origin. Define

$$\kappa = \sup_{x \in \overline{B(0,1/2)}} \mathbb{P}_x(\{\tau(\partial B(0,1)) < \tau(C_0)\})$$

.

Then $\kappa < 1$ and for any integers $m, n > 0$,

$$x, z \in \mathbb{R}^d,\ |x - z| < 2^{-n}m \Longrightarrow \mathbb{P}_x(\{\tau(\partial B(z,m)) < \tau(C_z)\}) \leq \kappa^n$$

Notation $\tau(\partial B(0,1)) : \Omega \to \tau(\partial B(0,1))(\omega) = \min\{t \geq 0 : B_t(\omega) \in \partial B(0,1)\} \in \mathbb{R}$ is the first time the Brownian motion hits the boundary of $B(0,1)$, and similary $\tau(C_0)$ is the first time the Brownian motion enters the cone.

Proof Since $0 < \frac{Leb(C_0)}{Leb(B(0,1)} < 1$, it is clear that $\kappa < 1$. Now consider the concentric balls $\{B(0,2^{-n})\}_{n=0}^{\infty}$.

We have $\{\tau(\partial B(0,1)) < \tau(C_0)\} \subset \bigcap_{n=0}^{\infty} \{\tau(\partial B(0,2^{-n})) < \tau(C_0)\}$. This is also a collection of independent sets, hence $\mathbb{P}_x(\{\tau(\partial B(0,1)) < \tau(C_0)\}) \leq$

$\Pi_{i=0}^{n-1}\mathbb{P}_x(\{\tau(\partial B(0,2^{-n+i+1})) < \tau(C_0)\}) \leq \kappa^n$. The more general formula is obtained by rescaling.

Theorem (Dirichlet Problem) Let $U \subset \mathbb{R}^d$ be a bounded, open, connected and such that every point on the boundary satisfies the Poincare cone condition. Let $\varphi : \partial U \to \mathbb{R}$ be continuous. Since U is bounded $\tau = \tau(\partial U) = \inf\{t \geq 0 : B_t \in \partial U\}$ is a finite stopping time. Then the function $u : \overline{U} \to \mathbb{R}$, $u(x) = \mathbb{E}_x[\varphi(B_\tau)]$ for $x \in \overline{U}$ is the unique continuous solution to the Dirichlet problem.

Proof The uniquess does not involve probabilistic arguments; it is the typical argument: if u_1 and u_2 are two solutions then applying the Maximum principle to their difference shows that they are equal.

The boundedness of U also assures us that u is bounded, as can directly be seen form its definition. Hence from the previous Proposition, u is harmonic.

Moreover, also straight form the definition, when $\mathbf{B}$ is a Brownian motion seeded at $x \in \partial U$, then $\tau(\partial U) = 0$ and so $\varphi(B(\tau(\partial U))) = \varphi(x)$ and so u also satisfies the boundary condition.

The rest of the proof consists in showing then that the Poincare condition implies that u is continuous on the boundary. Let $z \in \partial U$, C_z be a cone and $B(z,m)$ a ball so that the Poincare condition is satisfied at z. By the previous Lemma we have for all $x \in B(z, m2^{-n})$, $\mathbb{P}_x(\{\tau(\partial B(z,m)) < \tau(C_z)\}) \leq \kappa^n$.

Now by continuity of φ, given $\epsilon > 0$ choose $\delta \in (0,m)$ such that $|\varphi(y) - \varphi(z)| < \epsilon$ whenever $y \in \partial U \cap B(z, m2^{-n})$. If $x \in B(z, m2^{-n})$, then

$$\begin{aligned}|u(x) - u(z)| &= |\mathbb{E}_x[\varphi(B_\tau)] - \varphi(z)| \\ &\leq \mathbb{E}_x[|\varphi(B_\tau) - \varphi(z)|] \\ &= \int_{\{\tau(\partial U)<\tau(\partial B(z,m))\}} |\varphi(B_\tau) - \varphi(z)| d\mathbb{P} + \int_{\{\tau(\partial U)>\tau(\partial B(z,m))\}} |\varphi(B_\tau) - \varphi(z)| d\mathbb{P}\end{aligned}$$

If the Brownian motion hits the cone C_z before the sphere $\partial B(z,m)$, i.e. $\{\tau(\partial U) < \tau(\partial B(z,m))\}$, then $|\varphi(B_\tau) - \varphi(z)| < \epsilon$.

Then $|u(x) - u(z)| \leq \epsilon\mathbb{P}(\{\tau(\partial U) < \tau(\partial B(z,m))\}) + 2\|\varphi\|_\infty \mathbb{P}(\{\tau(\partial U) > \tau(\partial B(z,m))\}) \leq \epsilon + 2\|\varphi\|_\infty \kappa^n$. Taking $n \to \infty$ and $\epsilon \to 0$ gives continuity of φ.

3 The Heat Equation

3.1 Transition Kernels

Let $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}^d$. A function $p : [0,\infty) \times \mathbb{R}^d \times \mathcal{B} \to \mathbb{R}$ is called a Markov transition kernel if

1. $p(\cdot,\cdot,A)$ is a measurable function for all $A \in \mathcal{B}$.
2. $p(t,x,\cdot)$ is a Borel probability measure on $\mathbb{R}^d$ for all $t \geq 0$ and $x \in \mathbb{R}^d$. When integrating against this measure we write $\int f(\xi)p(t,x,d\xi)$ instead of $\int f(\xi)dp(t,x,\cdot)\xi$

3. For all $A \in \mathcal{B}$, $x \in \mathbb{R}^d$ and $t, s > 0$:

$$p(t+s, x, A) = \int_{\mathbb{R}^d} p(t, y, A) p(s, x, dy)$$

Intuitively, the Markov transition kernel $p(t, x, A)$ gives the probability that the process currently at x takes a value in the set A at time t.

Straight from the axioms of Brownian motion, it is clear that the distribution of the probability measure $p(t, x, \cdot)$ is normal with mean x and variance t. Therefore the the density (Radon-Nikodym derivative with respect to Lebesgue measure) of the probability measure $p(t, x, \cdot)$ of a d-dimensional Brownian motion is given by $\frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x-y|^2}{2t}}$, which we shall denote by $p(t, x, y)$. Now this looks very similar to the fundamental solution of the heat equation, so we are on to something.

3.2 The Heat Equation

Theorem (Heat Equation) Let $f(x) \in C_b(\mathbb{R}^d)$. Then the function $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$, $u(t, x) = \mathbb{E}_x[f(B_t)]$ is a continuous solution to the heat equation $\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x)$, for $(t, x) \in (0, \infty) \times \mathbb{R}^d$, $\lim_{t \to 0} u(t, x) = f(x)$ for $x \in \mathbb{R}^d$.

Proof

We have

$$u(t, x) = \mathbb{E}_x[f(B_t)] = \int_\Omega f(B_t(\omega)) d\mathbb{P} = \int_{\mathbb{R}^d} f(y) p(t, x, y) dy$$

From here we see that u is continuous and an application of the dominated convergence theorem shows that $u(t, x) \to f(x)$ as $t \to 0$.

From the density of the probability measure $p(t, x, \cdot)$ we have $\frac{\partial p}{\partial t}(t, x, y) = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}(t, x, y)$ for every $y \in \mathbb{R}^d$. Hence another application of the dominated convergence theorem shows that

$$\begin{aligned}
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} f(y) p(t, x, y) dy &= \int_{\mathbb{R}^d} f(y) \frac{\partial p(t, x, y)}{\partial t} dy \\
&= \int_{\mathbb{R}^d} f(y) \frac{1}{2} \frac{\partial^2 p(t, x, y)}{\partial x^2} dy \\
&= \frac{1}{2} \frac{\partial^2}{\partial x^2} \int_{\mathbb{R}^d} f(y) p(t, x, y) dy
\end{aligned}$$

4 The Feynman-Kac Representation

4.1 Itô Integrals

Recall that for $\varphi : [0,\infty) \to \mathbb{R} \in C \cap BV_{loc}([0,\infty))$ and $\psi : [0,\infty) \to \mathbb{R}$, the Riemann-Stieltjes integral is defined as

$$\int_0^t \psi(s)d\varphi(s) = \lim_{|\mathcal{P}|\to 0} \sum_{I\in\mathcal{P}} \psi(x_I)\Delta\varphi_I$$

where $\mathcal{P}$ is a partition of $[0,t]$. If ψ is Riemann-Stieltjes integrable with respect to φ, then φ is Riemann-Stieltjes integrable with respect to ψ and the integration by parts formula holds:

$$\int_0^t \varphi(s)d\psi(s) = \varphi(t)\psi(t) - \varphi(0)\psi(0) - \int_0^t \psi(s)d\varphi(s)$$

Notice that the sample paths of Brownian motion are continuous, so in the above discussion we could let $\psi(s) = B_s(\omega)$ (for fixed ω). More generally, if $\theta : [0,\infty)\times\Omega \ni (t,\omega) \to \theta_t(\omega) \in \mathbb{R}$ is a continuous stochastic process such that for almost every $\omega \in \Omega$, $\theta_{(\cdot)}(\omega) \in C \cap BV_{loc}([0,\infty))$, then we define the stochastic process

$$\mathcal{I}_\theta : [0,\infty)\times\Omega \ni (t,\omega) \to \mathcal{I}_\theta(t,\omega) = \int_0^t \theta_\tau(\omega)dB_\tau(\omega)$$

.

Of course

$$\mathcal{I}_\theta(t,\omega) = \theta_t(\omega)B_t(\omega) - \int_0^t B_\tau(\omega)d\theta_\tau(\omega)$$

As it is understood that $\mathcal{I}_\theta$ is a stochastic process we do not write everywhere the ω, so we have

$$\mathcal{I}_\theta(t) = \int_0^t \theta_\tau dB_\tau$$

The Itô integral is a stochastic process which has many nice properties, and behaves similarly to Brownian motion. For example, it is a continuous square-integrable martingale. Importantly, we have:

Theorem (Itô's formula) Let $F : (t,x) \in [0,\infty)\times\mathbb{R} \to F(t,x) \in \mathbb{R}$ be such that $F_1 = \frac{\partial F}{\partial t}$, $F_2 = \frac{\partial F}{\partial x}$ and $F_{22} = \frac{\partial^2 F}{\partial x^2}$ are continuous. Suppose that θ is a stochastic as described above. Then for all $t \geq 0$, a.s.

$$F(t,\mathcal{I}_\theta(t)) = F(0,0)+\int_0^t F_1(\tau,I_\theta(\tau))d\tau+\int_0^t F_2(\tau,I_\theta(\tau))\theta_\tau dB_\tau+\frac{1}{2}\int_0^t F_{22}(\tau,I_\theta(\tau))\theta_\tau^2 d\tau$$

or in differential form:

$$dF(t,\mathcal{I}_\theta(t)) = F_1(t,I_\theta(t))dt + F_2(t,I_\theta(t))\theta_t dB_t + \frac{1}{2}F_{22}(t,I_\theta(t))\theta_t^2 dt$$

Starting with a function F we can easily derive a stochastic differential equation (SDE) by writing out Itô's formula whose solution is known.

4.2 The Feynman-Kac Representation

Theorem

Assume that $\{X_t\}_{t\geq 0}$ is a real-valued stochastic process seeded at $x \in \mathbb{R}$ that solves the following SDE

$$\begin{cases} dX_t = \sigma(X_t)dB_t + b(X_t)dt \\ \qquad X_o \equiv x \end{cases}$$

where $\sigma : \mathbb{R} \to \mathbb{R}^+$ and $b : \mathbb{R} \to \mathbb{R}$.

Suppuse further that $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is such that $u_1 = \frac{\partial u}{\partial t}$, $u_2 = \frac{\partial u}{\partial x}$ and $u_{22} = \frac{\partial^2 u}{\partial x^2}$ are continuous and u solves the PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\sigma(x)^2\frac{\partial^2 u}{\partial x^2}(t,x) + b(x)\frac{\partial u}{\partial x}(t,x) & \text{for}(t,x) \in (0,\infty)\times\mathbb{R} \\ \qquad \lim_{t\to 0} u(t,x) = f(x) & \text{for} x \in \mathbb{R} \end{cases}$$

where $f \in C_b(\mathbb{R})$. Then if u does not grow too fast,

$u(t,x) = \mathbb{E}_x[f(X_t)]$ for every $(t,x) \in (0,\infty)\times\mathbb{R}$.

Proof (sketch)

Consider the stochastic process $V : [0,t]\times\Omega \to \mathbb{R}$, $V_0 \equiv x$, and for $0 \leq s \leq t$, $V_s = u(t-s, X_s)$. We apply Itô's formula:

$$\begin{aligned} dV_s = du(t-s,X_s) &= -u_1(t-s,X_s)ds + u_2(t-s,X_s)dX_s + \frac{1}{2}u_{22}(t-s,X_s)(dX_s)^2 \\ &= -u_1(t-s,X_s)ds + u_2(t-s,X_s)(\sigma(X_s)dB_s + b(X_s)ds) + \\ &\frac{1}{2}u_{22}(t-s,X_s)(\sigma(X_s)dB_s + b(X_s)ds)^2 \\ &= \left(-u_1(t-s,X_s)ds + u_2(t-s,X_s) + \frac{1}{2}u_{22}(t-s,X_s)\sigma^2(X_s)\right)ds + \\ &u_2(t-s,X_s)\sigma(X_s)dB_s \\ &= u_2(t-s,X_s)\sigma(X_s)dB_s \end{aligned}$$

Therefore $V_s = \int_0^s u_2(t-s,X_s)\sigma(X_s)dB_s$, so V is an Itô integral and thus a martingale, hence the expectation value of the process V is constant with respect to s. In particular, $\mathbb{E}_x[u(t,X_0)] = \mathbb{E}_x[u(0,X_t)] \Leftrightarrow \mathbb{E}_x[u(t,x)] = \mathbb{E}_x[f(X_t)] \Leftrightarrow u(t,x) = \mathbb{E}_x[f(X_t)]$, as required.

References

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