MATH 581 ASSIGNMENT 6

DUE TUESDAY APRIL 16

1. Consider the Cauchy problem for the *Euler equations*

$$\partial_t u + P\mathrm{div}(u \otimes u) = 0,$$

where P is the Leray projector (we defined it in class). The initial datum g is divergence free and is in $H^s(\mathbb{R}^n)$ for some $s > \frac{n}{2} + 1$. We want to prove local well posedness in H^s, by approximating the Euler equations by the Navier-Stokes equations. The following is a rough roadmap and some details may have to be tweaked to work.

- Prove uniqueness in class $\mathscr{C}([0,T], H^s)$.
- Prove that there exists $T > 0$, depending only on the H^s-norm of g, in particular independent of $\varepsilon > 0$, such that the Navier-Stokes equations

$$\partial_t u + P\mathrm{div}(u \otimes u) = \varepsilon \Delta u,$$

admit a solution $u_\varepsilon \in \mathscr{C}([0,T], H^s)$, with uniformly bounded norms.
- By compactness arguments, construct a solution of the Euler equations satisfying $u \in L^\infty_T H^s \cap \mathscr{C}([0,T], H^\sigma)$ for any $\sigma < s$.
- Show that $u : [0,T] \to H^s$ is continuous when H^s is equipped with its weak topology. Then prove that indeed $u \in \mathscr{C}([0,T], H^s)$ by showing that the norm $\|u(t)\|_{H^s}$ varies continuously in t.

2. Let $\Omega \subset \mathbb{R}^n$ be a domain and define the bilinear form

$$a(u,v) = \int_\Omega \Delta u \Delta v, \qquad u, v \in H^2(\Omega).$$

Suppose that X is a closed subspace of $H^2(\Omega)$, with $H^2_0(\Omega) \subset X$, and consider the variational problem: Find $u \in X$ such that

$$a(u,v) = \langle f, v \rangle, \qquad \text{for all } v \in X,$$

where $f \in L^2(\Omega)$. We refer to Chapter 7 of Folland's book for more on this setting.

a) What are the natural boundary conditions for the variational problem if $X = H^2(\Omega)$?
b) What are the natural boundary conditions if $X = H^2(\Omega) \cap H^1_0(\Omega)$?
c) Show that a is not coercive over $H^2(\Omega)$.
d) Show that a is coercive over $H^2(\Omega) \cap H^1_0(\Omega)$.

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e) Find a pair (a, X) corresponding to the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ∂_ν is the outer normal derivative at the boundary $\partial\Omega$. Investigate the solvability of this boundary value problem.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the second order elliptic operator given in divergence form

$$Lu = -\sum_{j,k=1}^{n} \partial_j (A_{jk} \partial_k u) + \sum_{j=1}^{n} B_j \partial_j u + Cu,$$

with smooth coefficients: $A_{jk}, B_j, C \in C^\infty(\overline{\Omega}, \mathbb{R}^{m\times m})$. Here u is understood to be a vector function on Ω with m (real) components. Assume that L is *strongly elliptic*, i.e.,

$$\sum_{j,k=1}^{n} \xi_j \xi_k [\eta^T A_{jk}(x) \eta] \geq c|\xi|^2 |\eta|^2, \qquad \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^m,\ x \in \overline{\Omega},$$

for some constant $c > 0$. Formally integrating $\langle Lu, v\rangle$ by parts with $v \in \mathscr{D}(\Omega)$, we are led to the bilinear form

$$a(u, v) = \int_\Omega (\partial_j v)^T A_{jk} \partial_k u + v^T B_j \partial_j u + v^T C u,$$

where the summation convention is assumed. Prove the followings.

a) The bilinear form a is coercive in $H^1_0(\Omega)$, i.e., the Gårding inequality

$$a(u, u) \geq c\|u\|^2_{H^1} - c_1 \|u\|^2_{L^2}, \qquad u \in H^1_0(\Omega),$$

is valid for some constants $c > 0$ and $c_1 \geq 0$.

b) For $\lambda \in \mathbb{R}$ sufficiently large, and for $f \in L^2(\Omega)$, there exists a unique $u \in H^1_0(\Omega)$ such that $Lu + \lambda u = f$.

c) Interior regularity: If $Lu = f$ with $u \in H^1_0(\Omega)$ and $f \in H^s(\Omega)$ then $u \in H^{s+2}(U)$ for any open U with $\overline{U} \subset \Omega$.

d) Regularity up to the boundary: In the same setting, $u \in H^{s+2}(\Omega)$.

4. With reference to the preceding problem, in linear elasticity, one has $m = n$ and

$$Lu = -\mu\Delta - (\mu + \lambda)\nabla(\nabla \cdot u),$$

where the real constants μ and λ are called *Lamé coefficients*.

a) Show that L is strongly elliptic if and only if $\mu > 0$ and $2\mu + \lambda > 0$. Assume in the followings that the Lamé coefficients satisfy these conditions.

b) Prove that the bilinear form a corresponding to L is not only coercive, but also *strictly coercive* in $H^1_0(\Omega)$. This result is called *Korn's first inequality.*

c) Conclude that the equation $Lu = f$ has a unique solution $u \in H^1_0(\Omega)$ for each $f \in L^2(\Omega)$.