# MATH 581 ASSIGNMENT 4

DUE FRIDAY MARCH 15

1. For $s \in \mathbb{R}$, the (Bessel potential) *Sobolev space* $H^s(\mathbb{R}^n)$ is the set of those $u \in \mathscr{S}'(\mathbb{R}^n)$ with $\|u\|_{H^s} := \|\langle D\rangle^s u\|_{L^2} < \infty$, where the *Bessel potential* $\langle D\rangle^s u$ of $u$ is defined by
$$\widehat{\langle D\rangle^s u}(\xi) = \langle\xi\rangle^s \hat{u}(\xi) \equiv (1+|\xi|^2)^{s/2}\hat{u}(\xi).$$
Prove the followings.
a) $\langle D\rangle^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert space isometry.
b) For $k \geq 0$ integer, $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.
c) $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.
d) The (topological) dual of $H^s(\mathbb{R}^n)$ is isometric to $H^{-s}(\mathbb{R}^n)$.
2. Prove the followings.
a) If $s = \frac{n}{2} + k + \alpha$ with $0 < \alpha < 1$ and $k \geq 0$ an integer, then $H^s(\mathbb{R}^n) \hookrightarrow C^{k,\alpha}(\mathbb{R}^n)$.
b) The *trace operator* $\gamma : \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^{n-1})$ defined by
$$(\gamma u)(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{n-1}, 0),$$
has a unique extension to a bounded linear operator $\gamma : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.
c) If $u \in H^s(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$, then $\varphi u \in H^s(\mathbb{R}^n)$ with
$$\|\varphi u\|_{H^s} \leq C\|u\|_{H^s},$$
where
$$C = 2^{|s|/2}\int_{\mathbb{R}^n} \langle\xi\rangle^{|s|}|\hat{\varphi}(\xi)|\mathrm{d}\xi.$$
*Hint*: Verify *Peetre's inequality*
$$\langle\xi\rangle^{2s} \leq 2^{|s|}\langle\xi-\eta\rangle^{2|s|}\langle\eta\rangle^{2s},$$
for $\xi, \eta \in \mathbb{R}^n$ and $s \in \mathbb{R}$.
d) Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with $\mathrm{d}\phi \in W^{\ell,\infty}(\mathbb{R}^n)$ and $\mathrm{d}(\phi^{-1}) \in W^{\ell,\infty}(\mathbb{R}^n)$ for all $\ell$. Then the pullback $\phi^* : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is a linear homeomorphism.
3. In this exercise we will refine the hypoellipticity and microlocal regularity theorems we have seen in class. First we localize the notion of Sobolev regularity in space and frequency. With $\Omega \subset \mathbb{R}^n$ a domain, let $u \in \mathscr{D}'(\Omega)$, and let $(x_0, \xi_0) \in \Omega \times \mathbb{R}^\times_n$, where $\mathbb{R}^\times_n = \mathbb{R}^n \setminus \{0\}$. We write $u \in H^s(x_0)$ if there is $\phi \in \mathscr{D}(\Omega)$ with $\phi(x_0) \neq 0$ such that $\phi u \in H^s(\mathbb{R}^n)$. Similarly, we write $u \in H^s(x_0, \xi_0)$ if there is $\phi \in \mathscr{D}(\Omega)$ with $\phi(x_0) \neq 0$,

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and there is a conical neighbourhood $V$ of $\xi_0$ such that $\langle\cdot\rangle^s\widehat{\phi u}$ is square integrable in $V$. With these localizations at hand, we define the singular support

$$\operatorname{sing\,supp}^s(u) = \Omega \setminus \{x \in \Omega : u \in H^s(x)\},$$

and the wave front set

$$\mathrm{WF}^s(u) = \Omega \times \mathbb{R}^\times_n \setminus \{(x,\xi) : u \in H^s(x,\xi)\},$$

adapted to Sobolev regularity. Prove one of the followings[1].

a) Let $p$ be a polynomial of degree $m$ in $\mathbb{R}^n$, satisfying

$$|\xi|^\gamma \lesssim \mu_p(\xi), \qquad \xi \in \mathbb{R}^n,$$

for some constant $0 < \gamma \leq 1$, where

$$\mu_p(\xi) = \inf\{|\eta| : p(\xi + i\eta) = 0,\, \eta \in \mathbb{R}^n\}.$$

Then

$$\operatorname{sing\,supp}^{s+\gamma m}(u) \subset \operatorname{sing\,supp}^s(p(D)u), \qquad u \in \mathscr{D}'(\Omega).$$

b) Let $P$ be a differential operator of order $m$ with smooth coefficients in $\Omega$. Then

$$\mathrm{WF}^{s+m}(u) \subset \mathrm{Char}P \bigcup \mathrm{WF}^s(Pu), \qquad u \in \mathscr{D}'(\Omega).$$

4. For a domain $\Omega \subset \mathbb{R}^n$, we define

$$H^s(\Omega) = \{u \in \mathscr{D}'(\Omega) : u = w|_\Omega \text{ for some } w \in H^s(\mathbb{R}^n)\},$$

with the norm

$$\|u\|_{H^s(\Omega)} = \inf_{\{w \in H^s(\mathbb{R}^n) : w|_\Omega = u\}} \|w\|_{H^s}.$$

Similarly, define

$$\mathscr{D}(\overline{\Omega}) = \{u : u = w|_\Omega \text{ for some } w \in \mathscr{D}(\mathbb{R}^n)\}.$$

a) Show that the restriction operator $w \mapsto w|_\Omega : H^s(\mathbb{R}^n) \to H^s(\Omega)$ is continuous, and that $\mathscr{D}(\overline{\Omega})$ is dense in $H^s(\Omega)$.

b) Show that there exists a sequence $\{\lambda_k\}$ satisfying

$$\sum_{k=0}^{\infty} 2^{jk}\lambda_k = (-1)^j, \qquad j \in \mathbb{N}_0.$$

c) Define the *Seeley extension operator* $E : \mathscr{D}(\overline{\mathbb{R}^n_+}) \to \mathscr{D}(\mathbb{R}^n)$ by

$$(Eu)(x) = \begin{cases} u(x) & \text{if } x_n \geq 0, \\ \sum_{k=0}^{\infty} \lambda_k u(x_1, \ldots, x_{n-1}, -2^k x_n) & \text{if } x_n < 0. \end{cases}$$

Prove that indeed $E$ maps $\mathscr{D}(\overline{\mathbb{R}^n_+})$ into $\mathscr{D}(\mathbb{R}^n)$, and that $E : H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n)$ is bounded for $s \geq 0$.

[1] If you prove both, you will get bonus 2 points towards your final grade

d) Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. By using coordinate transformations and partitions of unity, construct a bounded extension operator $E : H^s(\Omega) \to H^s(\mathbb{R}^n)$ for $s \geq 0$.

5. Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. Prove the followings.

a) If $s = \frac{n}{2} + k + \alpha$ with $0 < \alpha < 1$ and $k \geq 0$ an integer, then $H^s(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$.

b) If $\Omega$ is bounded and $s > t \geq 0$, then the embedding $H^s(\Omega) \hookrightarrow H^t(\Omega)$ is compact. You can use the fact that the embedding $H_0^s(U) \hookrightarrow H_0^t(U)$ is compact for bounded domains $U$, where $H_0^s(U)$ is the closure of $\mathscr{D}(U)$ with respect to the $H^s$ norm.

c) Let $\{U_k\}$ be a finite open cover of a neighbourhood of $\Omega$, and let $\{\varphi_k\}$ be a smooth partition of unity subordinate to $\{U_k\}$. Then

$$\|u\|_{H^s(\Omega)}^2 \eqsim \sum_k \|\varphi_k u\|_{H^s(U_k \cap \Omega)}^2, \qquad \text{for} \quad u \in H^s(\Omega).$$

In particular, the membership $u \in H^s(\Omega)$ is equivalent to $\varphi_k u \in H^s(U_k \cap \Omega)$ $\forall k$.