MATH 581 ASSIGNMENT 3

DUE FRIDAY MARCH 1

1. a) Derive a formula for $\widehat{u \circ A}$, where A is an $n \times n$ invertible matrix.
   b) There are (at least) two ways to define the Fourier transform on $L^2(\mathbb{R}^n)$.
      - Extend the Fourier transform from $\mathscr{S}$ to L^2 by using the density of $\mathscr{S}$ in L^2 (as well as the Plancherel bound).
      - First define the Fourier transform on $\mathscr{S}'$ by duality, and then restrict it to L^2.

      Show that these two approaches are consistent with each other.
   c) Show that the Fourier transform acting on L^1 is not onto $\mathscr{C}_0$.
   d) Give an example of $u \in C(\mathbb{R}^n)$ such that $\varphi \mapsto \int u\varphi$ is a tempered distribution and that there is no polynomial p satisfying $|u(x)| \leq |p(x)|$ for all $x \in \mathbb{R}^n$.
2. For each of the following functions, determine if it is a tempered distribution, and if so compute its Fourier transform.
   a) $x \sin x$,
   b) $\frac{1}{x} \sin x$,
   c) $e^{i|x|^2}$,
   d) $x\vartheta(x)$, where ϑ is the Heaviside step function,
   e) $\operatorname{sgn}(x) = \vartheta(x) - \vartheta(-x)$.
3. Prove that a distribution $u \in \mathscr{D}'$ is tempered if and only if $u = \partial^\alpha f$ for some continuous function f satisfying $|f(x)| \leq C(1 + |x|)^m$ with some constants C and m. That is, tempered distributions are derivatives of functions of polynomial growth.
4. a) Let $a \in \mathscr{E}(\mathbb{R}^n)$. Prove that the pointwise multiplication $u \mapsto au : \mathscr{S}' \to \mathscr{S}'$ is well-defined and continuous if and only if $a \in \mathscr{O}_M$, that is, for every multi-index α there is a polynomial p such that $|\partial^\alpha a(x)| \leq p(x)$, $x \in \mathbb{R}^n$.
   b) Prove that if p is a polynomial with no real zeroes, then there are constants $c > 0$ and m such that $|p(\xi)| \geq c(1 + |\xi|)^m$ for all $\xi \in \mathbb{R}^n$. Operators $p(D)$ with p satisfying this condition are called *strictly elliptic*.
   c) Show that if $p(D)$ strictly elliptic, then the equation $p(D)u = f$ has a solution for each $f \in \mathscr{S}'$.
5. Prove the followings.
   a) For a compactly supported distribution $u \in \mathscr{E}'$, its Fourier transform is equal to

$$\hat{u}(\xi) = \langle u(x), e^{-i\xi \cdot x} \rangle,$$

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where the notation $u(x)$ is to indicate that the distribution u acts on $e^{-i\xi\cdot x}$ as a function of x. The above expression also makes sense for $\xi \in \mathbb{C}^n$, defining an entire analytic function $\hat{u}$. (This is called the *Fourier-Laplace transform* of u.)

b) The *Paley-Wiener-Schwartz theorem*: Let $K \subset \mathbb{R}^n$ be a compact convex set, and let $\psi \in \mathscr{S}'$. Then a necessary and sufficient condition for ψ to be the Fourier transform of a distribution supported in K is that ψ is entire and satisfies the growth estimate

$$|\psi(\zeta)| \leq C(1+|\zeta|)^N e^{I_K(\eta)}, \qquad \zeta = \xi + i\eta \in \mathbb{C}^n,$$

with some constants C and N. Hence the Fourier-Laplace transform of a compactly supported distribution is an entire function of growth order at most 1. Recall that the indicator function I_K is defined as

$$I_K(\eta) = \sup_{x\in K} \eta \cdot x.$$

c) If the set of real zeroes of p is bounded, then every tempered distribution solution of $p(D)u = 0$ is an entire function of growth order at most 1.

6. Let p be a nonzero polynomial. Show the followings.
   a) The equation $p(D)u = f$ has at least one smooth solution for every $f \in \mathscr{D}$.
   b) If all solutions of $p(D)u = 0$ are smooth, then $\operatorname{sing\,supp} u \subset \operatorname{sing\,supp} p(D)u$ for any $u \in \mathscr{D}'$. So hypoelliptic operators can be defined as those $p(D)$ such that all solutions of $p(D)u = 0$ are smooth.
   c) If $p(D)$ admits a fundamental solution that is smooth outside some ball of finite radius (centred at the origin), then $p(D)$ is hypoelliptic.
7. Recall that by Hörmander's theorem, $p(D)$ is hypoelliptic if and only if for any $\eta \in \mathbb{R}^n$ one has $p(\xi + i\eta) \neq 0$ for all sufficiently large $\xi \in \mathbb{R}$.
   a) Construct a non-hypoelliptic polynomial p in dimension $n > 1$ such that $|p(\xi)| \to \infty$ as $|\xi| \to \infty$ for $\xi \in \mathbb{R}^n$.
   b) For any given $c > 0$, construct a non-hypoelliptic polynomial p in dimension $n > 1$ such that $|p(\xi + i\eta)| \to \infty$ uniformly in $\{|\eta| \leq c\}$ as $|\xi| \to \infty$ for $\xi \in \mathbb{R}^n$.