MATH 581 ASSIGNMENT 2

DUE FRIDAY FEBRUARY 15

1. If u is the characteristic function of the unit ball in $\mathbb{R}^n$, compute $x \cdot \nabla u$.
2. For $u \in \mathscr{E}'$ and $v \in L^1_{\text{loc}}(\mathbb{R}^n)$, we defined $u * v \in \mathscr{D}'$ by
$$\langle u * v, \varphi \rangle = \langle u, \tilde{v} * \varphi \rangle, \qquad \varphi \in \mathscr{D}.$$
Recall also the notations $\tilde{v}(z) = v(-z)$ and $(\tau_x \phi)(y) = \phi(y - x)$.
a) Show that if $u \in \mathscr{E}'$ and $v \in \mathscr{E}$ then $u * v \in \mathscr{E}$ and $(u * v)(x) = u(\tau_x \tilde{v})$ for $x \in \mathbb{R}^n$.
b) Show that for fixed $u \in \mathscr{E}'$, the mapping $v \mapsto u * v : \mathscr{E} \to \mathscr{E}$ is continuous.
3. *a)* Let $u \in \mathscr{E}'$ and $v \in \mathscr{D}'$. Show that
$$\tau_a(u * v) = (\tau_a u) * v = u * (\tau_a v), \qquad a \in \mathbb{R}^n,$$
where for distributions, the translation is defined by
$$\langle \tau_a u, \varphi \rangle = \langle u, \tau_{-a} \varphi \rangle.$$
b) For any distribution u, show that
$$\tau_a u = \delta_a * u,$$
where δ_a is the Dirac mass concentrated at $a \in \mathbb{R}^n$.
4. Let $u \in \mathscr{E}'$, $\phi \in \mathscr{E}$, and $\psi \in \mathscr{D}$. Prove that
$$u * (\phi * \psi) = (u * \phi) * \psi = (u * \psi) * \phi.$$
Show that
$$1 * (\delta' * \vartheta) \neq (1 * \delta') * \vartheta,$$
where 1 is the function identically 1 in $\mathbb{R}$, and ϑ is the Heaviside step function.
5. Let u and v be the surface measures of the spheres $\{x \in \mathbb{R}^3 : |x| = a\}$ and $\{|x| = b\}$, respectively. Compute $u * v$, and determine its singular support.
6. *a)* Find a fundamental solution of the heat operator $\partial_n - \sum_1^{n-1} \partial_j^2$.
b) Find a fundamental solution of the wave operator $\partial_n^2 - \sum_1^{n-1} \partial_j^2$, when $n = 2, 4$.
c) Find a fundamental solution of ∂^α with support in $\{x \in \mathbb{R}^n : x_j \geq 0,\ j = 1, \ldots, n\}$, where $\alpha_j \geq 1$, $j = 1, \ldots, n$.
d) Are these operators hypoelliptic?
7. Let $\omega \subset \mathbb{R}$ be an open interval, and $u \in \mathscr{D}'(\omega)$. Let $0 \leq k \leq \infty$.
a) Show that if $u' = 0$ then u is a constant function.
b) Show that if $u' = f$ with $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.
c) Show that if $u' + au = f$ with $a \in C^\infty(\omega)$ and $f \in C^k(\omega)$, then $u \in C^{k+1}(\omega)$.

Date: Winter 2013.