Some Topics in Semiclassical Analysis

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1 Introduction

The purpose of this project is to develop some of the basic theory of semiclassical microlocal analysis. Semiclassical analysis is motivated in large part by quantum mechanics. It is formulated in order to understand the relationships between dynamical systems and the behaviour of solutions of linear partial differential equations containing a small positive parameter $\hbar$.

A fundamental motivating question is how does classical dynamics determine the behaviour as $\hbar \to 0$ of Schrodinger's equation

$$i\hbar\partial_t u = -\hbar^2 \Delta u + V(x)u,$$

and the corresponding eigenvalue problem

$$-\hbar^2 \Delta u + V(x)u = Eu.$$

In this project, we are mostly concerned with the converse to this question, that is to say given mathematical objects associated with classical mechanics (i.e. classical observables), what is a reasonable and useful way to "quantize" them?

It should be noted that the techniques of semiclassical analysis apply in other settings and for other types of partial differential equations, but we will not have time to discuss them here.

The basic layout of this project is as follows. We first introduce the semiclassical Fourier transform, a generalisation of the classical Fourier transform that includes dependence on the small parameter $\hbar$. We then move on to stationary phase asymptotics, a critical technique for understanding the types of integrals that one deals with in this subject. Next we discuss various methods of quantizing our "symbols" (to be made more precise later, basically classical observables), the most important of which is the Weyl quantization. In the next section we prove formulae for the composition of operators. The penultimate step is to generalize our symbols classes so that we may do more refined analysis.

Finally, with applications in mind, we build operators on L^2 rather than $\mathcal{S}$. We finish by proving the main results of the paper, the weak and sharp Gårding inequalities.

It should be noted that, unless stated otherwise, the material here is adapted from Evans and Zworski's *Lectures on semiclassical analysis* [EZ10].

2 The Semiclassical Fourier Transform

For use later on, we define here the *semiclassical Fourier transform* for $\hbar > 0$, acting on functions in the Schwartz class:

$$\hat{\phi}(\xi) = \mathcal{F}_\hbar \phi(\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\langle x,\xi\rangle}\phi(x)dx \tag{1}$$

where $\langle \cdot\,,\cdot\rangle$ represents the usual Euclidean inner product. The inverse is given by

$$\mathcal{F}_\hbar{}^{-1}\psi(x) := \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x,\xi\rangle}\psi(\xi)d\xi. \tag{2}$$

This semiclassical Fourier transform has analogues to properties of the usual transform. Indeed, we have the following:

Theorem 1. *The following properties hold:*

$$(\hbar D_\xi)^\alpha \mathcal{F}_\hbar \phi = \mathcal{F}_\hbar((-x)^\alpha \phi),$$

$$\mathcal{F}_\hbar((\hbar D_x)^\alpha \phi) = \xi^\alpha \mathcal{F}_\hbar \phi,$$

and the $\hbar$-Plancherel theorem:

$$||\phi||_{L^2} = \frac{1}{(2\pi\hbar)^{n/2}}\,||\mathcal{F}_\hbar\phi||_{L^2}\,.$$

An interesting theorem that can now be proven is the uncertainty principle, which gives information on the extent to which it is possible to localize calculations in the x and ξ variables. We refer the reader to the proof given in [EZ10].

Theorem 2. The Uncertainty principle *We have*

$$\frac{\hbar}{2}\,||f||_{L^2}\,||\mathcal{F}_\hbar g||_{L^2} \leq ||x_j f||_{L^2}\,||\xi_j \mathcal{F}_\hbar f||_{L^2}\,, \quad (j = 1, \cdots, n) \tag{3}$$

3 Stationary Phase Asymptotics

In this section we develop the techniques of stationary phase in order to better understand the right hand side of (1), which will be necessary for some applications later.

We define for $\hbar > 0$ the following oscillatory integral for $a \in C_c^\infty(\mathbb{R}^n)$ and $\phi \in C^\infty(\mathbb{R}^n)$:

$$I_\hbar = I_\hbar(a, \phi) := \int_{\mathbb{R}^n} e^{\frac{i\phi}{\hbar}} a dx. \tag{4}$$

Right away we may derive the following asymptotic estimate:

Lemma 1. *If $\partial\phi' \neq 0$ on $K := \operatorname{supp}(a)$, then*

$$I_\hbar = O(h^\infty) \text{ as } h \to 0.$$

To clarify notation, this means that we must show that for all $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that

$$|I_\hbar| \leq C_N \hbar^N \sum_{|\alpha| \leq N} \sup_{\mathbb{R}^n} |\partial^\alpha a|,$$

where C depends only on K and n.

Proof. Define the operator L as

$$L := \frac{\hbar}{i} \frac{1}{\phi'(x)} \partial_x.$$

Note that this is well defined for $x \in K$, since $\phi' \neq 0$ there. Furthermore,

$$L\left(\frac{e^{i\phi}}{\hbar}\right) = e^{\frac{i\phi}{\hbar}}.$$

We note that $L^N(e^{i\phi/\hbar}) = e^{i\phi/\hbar}$ for all $N \in \mathbb{N}$. Hence,

$$|I_\hbar| = \left| \int L^N \left(e^{\frac{i\phi}{\hbar}}\right) a dx \right| = \left| \int e^{i\phi/\hbar} (L^*)^N a dx \right|.$$

Since $a \in C^\infty$ we have that $L^* a = -\frac{\hbar}{i} \partial_x \left(\frac{a}{\phi'}\right)$ is of order $\hbar$. Hence, $|I_\hbar| \leq C_N \hbar^N$. ☐

A theorem from analysis, the Morse lemma, will be needed to prove the important theorem from this section, the stationary phase asymptotics. We will state the Morse lemma now. For a proof see, for example, [Hör85].

Theorem 3. *(Morse Lemma) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be smooth with a nondegenerate critical point at x_0 (i.e. $\partial\phi(x_0) = 0, \det \partial^2\phi(x_0) \neq 0$). Then there exist neighbourhoods U of 0 and V of x_0 and a diffeomorphism*

$$\kappa : V \to U$$

such that

$$(\phi \circ \kappa^{-1})(x) = \phi(x_0) + \frac{1}{2}(x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots x_r^2), \tag{5}$$

where r is the number of positive eigenvalues of $\partial^2\phi(x_0)$.

We are now ready to prove the stationary phase theorem. Note that stationary phase has been applied to many problems in mathematics and physics. Although the proof to be presented here is one of two proofs in [EZ10], some other good references for stationary phase and other asymptotic methods (including a generalization of stationary phase, the method of steepest descent) are [Erd55] and [Hör90].

Theorem 4. *(Stationary phase asymptotics) Let $a \in C_c^\infty(\mathbb{R}^n)$. Let $x_0 \in K = \operatorname{supp}(a)$ and*

$$\partial\phi(x_0) = 0, \det(x_0) \neq 0,$$

and that $\partial\phi(x) \neq 0$ on $K \backslash \{x_0\}$. Then there exist for $k = 0, 1, \ldots$ differential operators $A_{2k}(x, D)$ of order less than or equal to $2k$ such that for each N

$$\left| I_\hbar - \left(\sum_{k=0}^{N-1} A_{2k}(x, D) a(x_0) \hbar^{k+\frac{n}{2}} \right) e^{\frac{i}{\hbar}\phi(x_0)} \right| \leq C_N \hbar^{N+\frac{n}{2}} \sum_{0 \leq m \leq 2N+n+1} \sup_{\mathbb{R}^n} |\partial^\alpha a|. \tag{6}$$

and hence, in particular,

$$A_0 = (2\pi)^{n/2} |\det \partial^2\phi(x_0)|^{-1/2} e^{\frac{i\pi}{4} \operatorname{sgn} \partial^2\phi(x_0)}, \tag{7}$$

which leads to the asymptotic estimate

$$I_\hbar = (2\pi\hbar)^{n/2} |\det \partial^2\phi(x_0)|^{-1/2} e^{\frac{i\pi}{4} \operatorname{sgn} \det^2 \phi(x_0)} e^{\frac{i\phi(x_0)}{\hbar}} a(x_0) + O(\hbar^{(n+2)/2}), \tag{8}$$

as $\hbar \to 0$.

Proof. Without loss of generality, we assume $x_0 = 0$ and $\phi(x_0) = 0$. After introducing a cutoff function χ, we apply Morse lemma, and the rapid decay lemma, lemma 1 to get

$$I_\hbar = \int_{\mathbb{R}^n} e^{\frac{i\phi}{\hbar}} a dx = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle Qx,x\rangle} u dx + O(h^\infty),$$

with

$$Q = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$$

□

and $u \in C^\infty$. In Q the upper identity matrix is $r \times r$ and the lower identitiy matrix is $(n-r) \times (n-r)$. Using some straightforward Fourier transform computations (for the details of which see the Fourier transform chapters of [EZ10]), we get

$$I_\hbar = \left(\frac{\hbar}{2\pi}\right)^{n/2} e^{\frac{i\pi}{4}} \operatorname{sgn} Q \int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\left\langle Q^{-1}\xi,\xi\right\rangle} \hat{u}(\xi) d\xi.$$

Setting:

$$J(\hbar, u) := \int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\left\langle Q^{-1}\xi,\xi\right\rangle} \hat{u}(\xi) d\xi;$$

then

$$\partial_\hbar J(\hbar, u) = \int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\left\langle Q^{-1}\xi,\xi\right\rangle} \left(-\frac{i}{2}\left\langle Q^{-1}\xi,\xi\right\rangle \hat{u}(\xi)\right) d\xi = J(\hbar, Pu),$$

where

$$P := -\frac{i}{2}\left\langle Q^{-1} D_x, D_x\right\rangle.$$

We hence get,

$$J(\hbar, u) = \sum_{k=0}^{N-1} \frac{\hbar^k}{k!} J(0, P^k u) + \frac{\hbar^N}{N!} R_N(\hbar, u),$$

with the remainder term

$$R_N(\hbar, u) := N \int_0^1 (1-t)^{N-1} J(t\hbar, P^N u) dt.$$

Then using the standard Fourier estimate $||\hat{u}||_{L^1} \leq C \sup_{|\alpha|\leq n+1} ||\partial^\alpha u||_{L^1}$, we get

$$|R_N| \leq C_N \left|\left|\widehat{P^N} u\right|\right|_{L^1} \leq C_N \sup_{|\alpha|\leq 2N+n+1} |\partial^\alpha a|.$$

In the next sections we will be mainly interested in the particular phase $\phi(x, y) = \langle x, y\rangle$.

4 Quantization formulas

In the last section we developed some theory concerning the semiclassical Fourier transform, which allows us to move between x and ξ variables. It is, however, desirable to be able to work with both sets of variables simultaneously. We will associate to "symbols" (this term will be made more precise soon) operators via various quantization schemes, and the resulting operators applied to functions can give information in the full (x, ξ) space, allowing us to do things such as localization in phase space.

After introducing quantization, one needs to work out the resulting symbol calculus, i.e. the rules for manipulating symbols and their associated operators.

4.1 Quantization schemes

For the time being, we shall call any function $a \in \mathcal{S} = \mathcal{S}(\mathbb{R}^{2n}), a = a(x, \xi)$ a *symbol*.

We begin by defining a very useful quantization, called the *Weyl quantization*. We define the Weyl quantization of a symbol a, denoted as $a^w(x, \hbar D)$ by its action on $u \in \mathcal{S}(\mathbb{R}^n)$:

$$a^w(x, \hbar D)u(x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} a\left(\frac{x+y}{2}, \xi\right) u(y)dyd\xi \tag{9}$$

This *standard quantization* of a is given by:

$$a(x, \hbar D)u(x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} a(x, \xi)u(y)dyd\xi \tag{10}$$

Most generally, for any $t \in [0, 1]$ we define:

$$\mathrm{Op}_t(a)u(x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} a(tx + (1-t)y, \xi)u(y)dyd\xi \tag{11}$$

Sometimes it is convenient to write $\mathrm{Op}(a)$ for $\mathrm{Op}_{1/2}(a)$, and hence $\mathrm{Op}(a) = a^w(x, \hbar D)$. Later on, in section 7,I we shall define another useful quantization, called the *anti-wick quantization*, which we shall use to prove the semiclassical sharp Gårding inequality.

As an example, we shall calculate $\mathrm{Op}_t(a)$ for the symbol $a(x,\xi) = \xi^\alpha$ for some multiindex α. We observe,

$$\begin{aligned}
\mathrm{Op}_t(a)u(x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(tx+(1-t)y,\xi)e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}u(y)dyd\xi \\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \xi^\alpha e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}u(y)dyd\xi \\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \xi^\alpha e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}u(y)d\xi dy \qquad (12) \\
&= \int_{\mathbb{R}^n} \mathcal{F}_\hbar^{-1}(\xi^\alpha)(x-y)u(y)dy \\
&= \int_{\mathbb{R}^n} (\hbar D)^\alpha \delta(x-y)u(y)dy \\
&= (\hbar D)^\alpha u(x)
\end{aligned}$$

where the interchange of the order of integration on line (12) is justified because $u\xi^\alpha$ is a Schwartz function, and so the integrand is absolutely integrable. Hence, we observe that $\mathrm{Op}_t(a) = (\hbar D)^\alpha$.

The following theorems give some important facts about this quantization scheme:

Theorem 5. *If $a \in \mathcal{S}$, then $\mathrm{Op}_t(a)$ can be defined as an operator that maps $\mathcal{S}'$ to $\mathcal{S}$, and the mapping $\mathrm{Op}_t(a) : \mathcal{S}' \to \mathcal{S}$, for $0 \le t \le 1$ is continuous.*

Theorem 6. *Let $a \in \mathcal{S}$. Then, for $0 \le t \le 1$ we have $\mathrm{Op}_t(a)^* = \mathrm{Op}_{1-t}(\overline{a})$ and hence if a is real we have $a^w(x,\hbar D)^* = a^w(x,\hbar D)$.*

Theorem 7. *If $a \in \mathcal{S}'$, then $\mathrm{Op}_t(a)$ can be defined as an operator that maps $\mathcal{S}$ to $\mathcal{S}'$, and the mapping $\mathrm{Op}_t(a) : \mathcal{S} \to \mathcal{S}'$ is continuous.*

5 Composition of operators

In this section we will discuss the problem of composition of operators. That is, to show that if a and b are symbols, then there exists a symbol c, such that

$a^w(x,\hbar D) \circ b^w(x,\hbar D) = c^w(x,\hbar D)$ and we write $c = a\#b$. We first consider linear symbols. Consider the following theorem

Theorem 8. *Let $(x^*,\xi^*) \in \mathbb{R}^{2n}$ and define the linear symbol*

$$l(x,\xi) := \langle x^*, x\rangle + \langle \xi^*, \xi\rangle\,. \tag{13}$$

Then,

$$\mathrm{Op}_t(l)u = \langle x^*, x\rangle\, u + \langle \xi^*, \hbar D, u\rangle \quad (0 \le t \le 1). \tag{14}$$

Proof. We begin by showing that $\mathrm{Op}_t(l)$ does not depend on t:

$$\begin{aligned}
\frac{d}{dt}\mathrm{Op}_t(l)u &= \frac{1}{(2\pi\hbar)^n}\frac{d}{dt}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}(\langle x^*, tx+(1-t)y\rangle + \langle \xi^*,\xi\rangle)u(y)dyd\xi\\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}\langle x^*, x-y\rangle\, u(y)dyd\xi\\
&= \frac{\hbar}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\left\langle x^*, D_\xi \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} u(y)dy\right\rangle d\xi\\
&= \frac{\hbar}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\left\langle x^*, D_\xi (e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}\hat{u}(\xi))\right\rangle d\xi.
\end{aligned}$$

Now, since $\hat{u}(\xi)$ decays rapidly as $|\xi| \to \infty$ the last expression vanishes, and hence we see that indeed $\mathrm{Op}_t(l)$ is independent of t. So for all $0 \le t \le 1$, $\mathrm{Op}_t(l)u = \mathrm{Op}_1(l)u = \langle x^*, x\rangle\, u + \langle \xi^*, \hbar D\rangle\, u$. ∎

Having proven this result, we will naturally just write $l(x,\hbar D)$ for $l^w(x,\hbar D)$. We are now ready to prove the rule for composition with a linear symbol.

Theorem 9. *Let $b \in \mathcal{S}$. Then,*

$$l(x,\hbar D) b^w(x,\hbar D) = c^w(x,\hbar D), \tag{15}$$

where

$$c := lb + \frac{\hbar}{2i}\{l,b\}. \tag{16}$$

Recall that $\{\cdot,\cdot\}$ is the poisson bracket defined by

$$\{l,b\} = \langle \partial_\xi l, \partial_x b\rangle\,.$$

Proof. By the just proven theorem 8 we know that

$$l(x, \hbar D) = \langle x^*, x \rangle + \langle \xi^*, \hbar D \rangle .$$

By definition we have that

$$\langle x^*, x \rangle\, b^w(x, \hbar D)u = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle x^*, x \rangle\, e^{\frac{i}{\hbar}\langle x-y,\xi \rangle} b\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

The key observation is that

$$\frac{x-y}{2} e^{\frac{i}{\hbar}\langle x-y,\xi \rangle} = \frac{\hbar}{2i} \partial_\xi \left(e^{\frac{i}{\hbar}\langle x-y,\xi \rangle} \right).$$

Hence, integrating by parts gives

$$\begin{aligned} &\langle x^*, x \rangle b^w(x, \hbar D) = \\ &\qquad \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi \rangle} \left(\left\langle x^*, \frac{x-y}{2} \right\rangle b - \frac{\hbar}{2i} \langle x^*, \partial_\xi b \rangle \right) u(y) du d\xi. \end{aligned}$$

In addition to the above, we have

$$\begin{aligned} &\langle \xi^*, \hbar D_x \rangle b^w(x, \hbar D) \\ &\qquad = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle \xi^*, \hbar D_x \rangle \left(e^{\frac{i}{\hbar}\langle x-y,\xi \rangle} b\left(\frac{x+y}{2}, \xi\right)\right) u(y) dy d\xi \\ &\qquad = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi \rangle} \left(\langle \xi^*, \xi \rangle\, b + \frac{\hbar}{2i} \langle \xi^*, \partial_x b \rangle \right) u(y) dy d\xi. \end{aligned}$$

Adding these last equations gives

$$\begin{aligned} &l(x, \hbar D) b^w(x, \hbar D) \\ &\qquad = (\langle x^*, x \rangle + \langle \xi^*, \hbar D \rangle) b^w(x, \hbar D) \\ &\qquad = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\langle x-y,\xi \rangle} \\ &\qquad\qquad \left(\left(\langle \xi^*, \xi \rangle + \left\langle x^*, \frac{x+y}{2} \right\rangle \right) b + \frac{\hbar}{2i} \{l, b\} \right) u(y) dy d\xi \end{aligned}$$

proving the result. $\square$

It is now quick to show that $\mathrm{Op}\left(e^{-\frac{i}{\hbar}l}\right) = e^{-\frac{i}{\hbar}l(x,\hbar D)}$, and to prove the representation formula

$$a^w(x,\hbar D) = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^n} \hat{a}(l) e^{\frac{i}{\hbar}l(x,\hbar D)} dl \tag{17}$$

where we define

$$\hat{a}(l) := \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}l(x,\xi)} a(x,\xi) dx d\xi.$$

The results from this section give the necessary theory to prove the theorem for composition of Weyl quantization. We present the theorem here without proof. The proof is very long, and I refer the read to [EZ10] for the proof.

Theorem 10. *Let $a, b \in \mathcal{S}$. We then have*

$$a^w(x,\hbar D) \circ b^w(x,\hbar D) = c^w(x,\hbar D)$$

for the symbol

$$c = a\#b,$$

where

$$a\#b(x,\xi) := e^{\frac{i\hbar}{2}\sigma(D_x,D_\xi,D_y,D_\eta)}(a(x,\xi)b(y,\eta))|_{x=y,\xi=\eta}, \tag{18}$$

where $\sigma(D_x, D_\xi, D_y, D_\eta) := \langle D_\xi, D_y\rangle - \langle D_x, D_\eta\rangle$. We also have an integral representation formula

$$\begin{aligned} a\#b(x,\xi) & \\ &= \frac{1}{(\pi\hbar)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x+z,\xi+\zeta)b(x+y,\xi+\eta) \\ &\qquad\qquad e^{\frac{2i}{\hbar}\sigma(y,\eta;z,\zeta)} dy d\eta dz d\zeta, \end{aligned} \tag{19}$$

6 General symbol classes

We will now extend our symbol calculus to symbols $a = a(x,\xi,\hbar)$, depending on a parameter $\hbar$. We will need a few definitions to do this.

Definition 1. *A function $m : \mathbb{R}^{2n} \to (0,\infty)$ is called an* order function *if there exist constants C, N such that*

$$m(z) \le C\langle z-w\rangle^N m(w),$$

for all $w, z \in \mathbb{R}^{2n}$.

Some examples of order functions are $m(z) \equiv 1$ and $m(z) = \langle z \rangle$. We also note that if m_1, m_2 are order functions, then so is $m_1 m_2$.

Definition 2. *Given some order function m, on $\mathbb{R}^{2n}$ we define the corresponding symbol class $S(m)$ by*

$$S(m) := \{a \in C^\infty : \textit{for each multiindex } \alpha \textit{ there exists a constant } C_\alpha \textit{ so that } |\partial^\alpha a| \leq C_\alpha m \}. \tag{20}$$

We also define two other associated symbol classes, $S^k(m)$ and $S^k_\delta(m)$ by

$$S^k(m) := \{a \in C^\infty : |\partial^\alpha a| \leq C_\alpha \hbar^k m \textit{ for all multiindices } \alpha\} \tag{21}$$

$$S^k_\delta(m) := \{a \in C^\infty : |\partial^\alpha a| \leq C_\alpha \hbar^{-\delta|\alpha|-k} m \textit{ for all multiindices } \alpha\} \tag{22}$$

In the above definition, we see that k describes how singular the symbol a is as $\hbar \to 0$ and that δ allows for increasing singularity of higher derivatives. We also define the natural class of symbols $S^{-\infty}(m)$ by

$$S^{-\infty}(m) := \{a \in C^\infty : \text{for each } \alpha \text{ and } N,\, |\partial^\alpha a| \leq C_{\alpha,N} \hbar^N m\}$$

Hence, if a is a symbol belonging to $S^{-\infty}(m)$, then a and all of its derivatives are $O(\hbar^\infty)$ as $\hbar \to 0$.

In order to simplify notation, note that if the order function is the constant function $m \equiv 1$, we will just write $S^k := S^k(1)$ and $S^k_\delta := S^k_\delta(1)$. We will also omit zero superscripts.

We have many of the same results about quantization for these general symbol classes. We will now give a theorem about thw Weyl quantization for a symbol in the class $S_\delta(m)$.

Theorem 11. *Let $a \in S_\delta(m)$, where m is some order function. Then*

$$\mathrm{Op}(a) : \mathcal{S} \to \mathcal{S}.$$

Proof. We may, without loss of generality, rescale to $\hbar = 1$ (explanation on page 52 of [EZ10]). Let

$$\mathrm{Op}(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a\left(\frac{x+y}{2}, \xi\right) u(y) d\xi dy,$$

for $u \in \mathcal{S}$. Furthermore, we see that, for the operator

$$L_1 := \frac{1 + \langle x - y, D_\xi \rangle}{1 + \langle x - y \rangle^2}$$

we have that $L_1 e^{i\langle x-y,\xi\rangle} = e^{i\langle x-y,\xi\rangle}$. and for

$$L_2 := \frac{1 - \langle \xi, D_y \rangle}{1 + \langle \xi \rangle^2},$$

we have that $L_2 e^{i\langle x-y,\xi\rangle} = e^{i\langle x-y,\xi\rangle}$. Now, observe that

$$x_j \operatorname{Op}(a)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (D_{\xi_j} + y_j) e^{i\langle x-y,\xi\rangle} aud\xi dy.$$

By integrating by parts, we see that $x^\alpha \operatorname{Op}(a) : \mathcal{S} \to L^\infty$. Now since

$$\operatorname{Op}_t(a) \left(e^{-i(\frac{1}{2}-t)D_x D_\xi} a \right) = \operatorname{Op}(a),$$

we see that

$$\begin{aligned}
D_{x_j} \operatorname{Op}(a)u &= D_{x_j} \operatorname{Op}_0 \left(e^{-\frac{i}{2} D_x D_\xi} a \right) u \\
&= D_{x_j} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{i}{2} D_x D_\xi} a(y,\xi) e^{i\langle x-y,xi\rangle} u(y) d\xi dy \right) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{i}{2} D_x D_\xi} a(y,\xi)(-D_{y_j} e^{i\langle x-y,\xi\rangle}) u(y) d\xi dy.
\end{aligned}$$

Another integration by parts shows that $D^\beta \operatorname{Op}(a) : \mathcal{S} \to L^\infty$. These two estimates together show us that $D^\beta x^\alpha \operatorname{Op}(a) : \mathcal{S} \to L^\infty$ for all multiindices α, β, and hence $\operatorname{Op}(a) : \mathcal{S} \to \mathcal{S}$. $\square$

7 Operators on L^2

The analysis that has been developed so far has built operators on either $\mathcal{S}$ or its dual $\mathcal{S}'$. However, since L^2 is a very important space in applications, we now wish to develop some theory in the special case of functions in L^2. The ultimate goal of this section is to prove the semiclassical version of the sharp Gårding inequality, which like its classical counterpart is essential in the analysis of pseudodifferential operators.

For the time being we shall take $\hbar = 1$. Let $\chi \in C_c^\infty(\mathbb{R}^{2n})$ be such that

$$0 \leq \chi \leq 1, \quad \chi \equiv 0 \text{ on } \mathbb{R}^{2n} \backslash B(0,2),$$

and

$$\sum_{\alpha \in \mathbb{Z}^{2n}} \chi_\alpha \equiv 1,$$

where $\chi_\alpha := \chi(\cdot - \alpha)$ is χ shifted by the lattice point $\alpha \in \mathbb{Z}^{2n}$. We introduce the notation

$$a_\alpha := \chi_\alpha a;$$

and hence

$$a = \sum_{a \in \mathbb{Z}^2 n} a_\alpha.$$

In addition, we define

$$b_{\alpha\beta} := \overline{a}_\alpha \# a_\beta (\alpha, \beta \in \mathbb{Z}^{2n}).$$

We now present and prove the following estimate:

Theorem 12. *For each N and each multiindex γ, we have that*

$$|\partial^\gamma b_{\alpha\beta}(z)| \leq C_{\gamma,N} \langle \alpha - \beta \rangle^{-N} \langle z - \frac{\alpha + \beta}{2}^{-N} \rangle, \tag{23}$$

where $z = (x, \xi) \in R^{2n}$.

Proof. We have the following explicit formula:

$$b_{\alpha\beta}(z) = \frac{1}{\pi^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{i\phi(w_1,w_2)} \overline{a}_\alpha(z - w_1) \alpha_\beta(z - w_2) dw_1 dw_2,$$

where

$$\phi(w_1, w_2) = -2\sigma(x, \xi, y, \eta) = 2\langle x, \eta \rangle - 2\langle \xi, y \rangle,$$

and

$$w = (w_1, w_2), w_1 = (x, \xi), w_2 = (y, \eta).$$

Choosing $\zeta : \mathbb{R}^{4n} \to \mathbb{R}$ such that $0 \leq \zeta \leq 1, \zeta \equiv 1$ on $B(0,1)$, $\zeta \equiv 0$ on $\mathbb{R}^{4n} - B(0,2)$. We can then decompose $b_{\alpha\beta}(z)$ as follows

$$\begin{aligned} b_{\alpha\beta}(z) &= c_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi} \zeta(w) \overline{a}_\alpha(z - w_1) a_\beta(z - w_2) dw_1 dw_2 \\ &\quad + c_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi} (1 - \zeta(w)) \overline{a}_\alpha(z - w_1) a_\beta(z - w_2) dw_1 dw_2 \\ &=: A + B \end{aligned}$$

We will now estimate A and B separately. Clearly, for A we have

$$|A| \leq \iint_{\{|w|\leq 2\}} |\overline{a}_\alpha(z-w_1)||a_\beta(z-w_2)|dw_1dw_2.$$

But this integrand equals

$$\chi(z-w_1-\alpha)\chi(z-w_2-\beta)|a(z-w_1)||a(z-w_2)|$$

which vanishes unless

$$|z-w_1-\alpha| \leq 2$$

and also

$$|z-w_2-\beta| \leq 2.$$

This implies that

$$|\alpha-\beta| \leq 4+|w_1|+|w_2| \leq 8$$

and

$$\left|z-\frac{\alpha+\beta}{2}\right| \leq 4+|w_1|+|w_2| \leq 8.$$

This then gives

$$|A| \leq C_N\langle\alpha-\beta\rangle^{-N}\langle z-\frac{\alpha+\beta}{2}\rangle^{-N}$$

for all N. A similar computation shows

$$|\partial^\gamma A| \leq C_{N,\gamma}\langle\alpha-\beta\rangle^{-N}\langle z-\frac{\alpha+\beta}{2}\rangle^{-N}.$$

To estimate B we observe that

$$\partial\phi(w_1,w_2) = 2(\eta,-y,-\xi,x),$$

and hence

$$|\partial\phi(w)| = 2|w|.$$

We also have that, for the operator

$$L := \frac{\langle\partial\phi, D\rangle}{|\partial\phi|^2},$$

we have $Le^{i\phi} = e^{i\phi}$. By using a usual integration by parts argument coupled with the fact that the integrand in B vanishes for $|w| \geq 1$ we get the following estimate for B:

$$|B| \leq C_M \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \langle w \rangle^{-M} \overline{A}_\alpha(z - w_1) A_\beta(z - w_2) dw_1 dw_2$$

and that $\operatorname{supp} A_\alpha \subseteq B(\alpha, 2), \operatorname{supp} A_\beta \subseteq (\beta, 2)$. So the integrand will vanish unless

$$\frac{1}{c}\langle w \rangle \leq \langle \alpha - \beta \rangle, \langle z - \frac{\alpha+\beta}{2} \rangle \leq C\langle w \rangle.$$

Hence, for sufficiently large M we have

$$\begin{aligned}|B| &\leq C_M \langle \alpha - \beta \rangle^{-N} \langle z - \frac{\alpha+\beta}{2} \rangle^{-N} \iint \langle w \rangle^{2N-M} dw_1 dw_2 \\ &\leq C_M \langle \alpha - \beta \rangle^{-N} \langle z - \frac{\alpha+\beta}{2} \rangle^{-N}.\end{aligned}$$

Similarly,

$$|\partial^\gamma B| \leq C_{N,\gamma} \langle \alpha - \beta \rangle^{-N} \langle z - \frac{\alpha+\beta}{2} \rangle^{-N}.$$

□

Theorem 13. *(Operator norms) For each N,*

$$||\mathrm{Op}(b_{\alpha\beta})||_{L^2 \to L^2} \leq C_N \langle \alpha - \beta \rangle^{-N}.$$

Proof. Recall that we earlier demonstrated that

$$\mathrm{Op}(a) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \hat{a}(l) \, \mathrm{Op}(e^{il}) dl.$$

We also know that $\mathrm{Op}(e^{il})$ is unitary on L^2. Hence,

$$||\mathrm{Op}(a)||_{L^2 \to L^2} \leq C \int_{\mathbb{R}^{2n}} |\hat{a}(l)| dl.$$

Hence for $M > 2n$ we can make the estimate, using theorem 12:

$$\begin{aligned} ||O(b_{\alpha\beta})||_{L^2\to L^2} &\leq C \left|\left|\hat{b}_{\alpha\beta}\right|\right|_{L^1} \\ &\leq C \left|\left|\langle \xi \rangle^M \hat{b}_{\alpha\beta}\right|\right|_{L^\infty} \\ &\leq C \max_{|\gamma|\leq M} \left|\left|\widehat{D^\gamma b_{\alpha\beta}}\right|\right|_{L^\infty} \\ &\leq C \max_{|\gamma|\leq M} ||D^\gamma b_{\alpha\beta}||_{L^1} \\ &\leq C \sup_{|\gamma|\leq M} \left|\left|\langle z \rangle^M D^\gamma b_{\alpha\beta}\right|\right|_{L^1} \\ &\leq C\langle \alpha - \beta \rangle^{-N}. \end{aligned}$$

□

Now, we can prove the following very important estimate:

Theorem 14. *(Calderon-Vaillancourt) If $0 \leq \delta \leq 1/2$ and we have a symbol $a \in S_\delta$ then*

$$\mathrm{Op}(a) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is bounded. Moreover, we have the estimate

$$||\mathrm{Op}(a)||_{L^2\to L^2} \leq C \sum_{|\alpha|\leq M} |\partial^\alpha a|.$$

Proof. Since we have that

$$\mathrm{Op}(b_{\alpha\beta}) = A_\alpha^* A_\beta,$$

by the previous theorem on operator norms we have that

$$||A_\alpha^* A_\beta||_{L^2\to L^2} \leq C\langle \alpha - \beta \rangle^{-N}.$$

Hence,

$$\sup_\alpha \sum_\beta \left|\left|A_\alpha A_\beta^*\right|\right|^{1/2} \leq C \sum_\beta \langle \alpha - \beta \rangle^{-N/2} \leq C.$$

Similarly,

$$\sup_\alpha \sum_\beta ||A_\alpha^* A_\beta||^{1/2} \leq C.$$

The result now follows from the Cotlar-Stein theorem (theorem 6 in appendix B of [EZ10]). Note that it was necessary that $\delta \leq 1/2$ because it is for precisely these cases that we can take $\hbar = 1$ via scaling without loss of generality. $\square$

A corollary of this theorem is the following:

Theorem 15. *Let $a, b \in S_\delta$ for $0 \leq \delta < 1/2$. Then,*

$$||a^w b^w - (ab)^w||_{L^2 \to L^2} = O(\hbar^{1-2\delta}),$$

as $\hbar \to 0$.

For our purposes we have developed a sufficiently general theory for the quantizations of symbols a belonging to appropriate symbol classes. We will now prove some nice properties of elliptic, respectively non-negative, symbols cumulating in showing the easy, respectively sharp, semiclassical Gårding inequalities. First we need a basic theorem of elliptic symbols.

Definition 3. *A symbol a is said to be* elliptic *if there exists a constant $\gamma > 0$ such that*

$$|a| \geq \gamma > 0, \quad \text{on } \mathbb{R}^{2n}.$$

Before proving the invertability of elliptic symbols, we recall the following theorem from functional analysis:

Theorem 16. *Let X, Y be Banach spaces and suppose $A : X \to Y$ is a bounded linear operator. suppose there exists bounded linear operators $B_1, B_2 : Y \to X$ such that*

$$AB_1 = I + R_1 \text{ on } Y,$$

and

$$B_2 A = I + R_2 \text{ on } X,$$

where

$$||R_1|| < 1, ||R_2||_< 1.$$

Then A is invertible.

Theorem 17. *(Invertability of elliptic symbols). Let $a \in S_\delta$ for $0 \leq \delta < \frac{1}{2}$ and assume a to be elliptic. Then for some $\hbar_0 > 0$ we have that $\mathrm{Op}(a)^{-1}$ exists as a bounded linear operator on $L^2(\mathbb{R}^n)$, provided that $0 < \hbar_0$.*

Proof. Since a is nonvanishing we know that it is pointwise invertible. Let $b := \frac{1}{a}$. $b \in S_\delta$. We note that we can write

$$a\#b = 1 + r_1, \quad r_1 \in S_\delta^{2\delta-1}.$$

By the same token,

$$b\#a = 1 + r_2, \quad r_2 \in S_\delta^{2\delta-1}.$$

So, letting $A := \mathrm{Op}(a), B := \mathrm{Op}(b), R_1 := \mathrm{Op}(r_1), R_2 := \mathrm{Op}(r_2)$ we have the relations

$$A \cdot B = I + R_1,$$
$$B \cdot A = I + R_2,$$

and we have

$$||R_1||_{L^2 \to L^2} = O(\hbar^{1-2\delta}) \leq \frac{1}{2},$$
$$||R_2||_{L^2 \to L^2} = O(\hbar^{1-2\delta}) \leq \frac{1}{2},$$

when $0 < \hbar \leq \hbar_0$. We conclude that A has an approximate left and right inverses. Hence, by the above theorem we see that $A^-1 = \mathrm{Op}(a)^{-1}$ exists. $\square$

We are now ready to prove the Gårding inequalities. We start with the easier version for elliptic symbols.

Theorem 18. *("Easy" Gårding inequality) Assume $a = a(x, \xi)$ is a real-valued symbol in S and that*

$$a \geq \gamma > 0 \text{ on } \mathbb{R}^{2n}.$$

Then for all $\epsilon > 0$ there exists $\hbar_0 = \hbar_0(\epsilon) > 0$ such that

$$\langle a^w(x, \hbar D)u, u\rangle \geq (\gamma - \epsilon) \, ||u||_{L^2}^2$$

for all $0 < \hbar \leq \hbar_0, u \in C_c^\infty(\mathbb{R}^n)$.

Proof. We begin by showing that $(a - \lambda)^{-1} \in \mathcal{S}$ if $\lambda < \gamma - \epsilon$. Letting $b := (a - \lambda)^{-1}$, then

$$(a - \lambda)\#b = 1 + \frac{\hbar}{2i}\{a - \lambda, b\} + O(\hbar^2) = 1 + O(\hbar)^2$$

(here we used the fact that the poisson bracket term vanishes because b is a function of $a - \lambda$. Thus,

$$(a^w - \lambda) \circ b^w = I + O(\hbar^2)_{L^2 \to L^2}.$$

This means that b^w is an approximate inverse of $a^w - \lambda$. The same calculation, mutatis mutandis, shows that b^w is also an approximate left inverse. So, by the approximate inverse theorem 16 we know that $a^w - \lambda$ is invertible for each $\lambda < \gamma - \epsilon$. As a consequence, we have

$$\mathrm{spec}(a^w) \subset [\gamma - \epsilon, \infty).$$

Then, by the spectral theory of self-adjoint operators (see, for example, appendix B theorem 1 in [EZ10]) we have

$$\langle a^w u, u \rangle \geq (\gamma - \epsilon) \, ||u||_{L^2}^2$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. □

We are now ready to prove our main result, the sharp semiclassical Gårding inequality. To my knowledge, there are at least two proofs of this theorem. The one given in [EZ10] uses a gradient estimate and the symbol calculus developed in this paper to derive the inequality. It is not that proof that shall be given here, however. The proof that shall be presented here introduces a new, but related quantization, the so called *(semiclassical) anti-Wick quantization*, $\mathrm{Op}^{aw}(a)$, of a symbol a. This quantization is useful in other semiclasscal results, and so we use the desire to prove the sharp Gårding inequality as a good excuse to introduce it. The outline of the proof is taken in part from [Mar02] and in part from [Bro].

A natural question that can be asked regarding these quantization schemes is the following: given a symbol $a \geq 0$, is it true that $\mathrm{Op}(f)$ is positive-definite? For the quantizations developed so far, the answer to this is no. However, we shall now introduce a quantization for which this property holds, and it is precisely this property that we later explot to prove the sharp Gårding inequality.

The goal is to prove the following theorem:

Theorem 19. *Let $a \in S^0(1)$ and $a(x, \xi, \hbar) \geq 0$. Then there exists $C, \hbar_0 > 0$ such that for $\hbar \in [0, \hbar_0]$,*

$$\langle a^w(x, \hbar D_x) u, u \rangle \geq -C\hbar \, ||u||_{L^2}^2 .$$

I will break down the proof of the theorem into several steps. We begin by defining, for $a \in S^0(1)$,

$$\tilde{a} := \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(y,\eta) e^{-(|x-y|^2+|\xi-\eta|^2)/\hbar} dy d\eta.$$

and we observe that $\tilde{a}(x,\xi) = a * G(x,\xi)$ as a convolution in $\mathbb{R}^{2n}$ where $G = e^{-(|x|^2+|\xi|^2)}$.

Lemma 2. *For $a \in S^0(1)$, we have that $\tilde{a}$ is bounded, and in fact $\tilde{a} \in S^0(1)$.*

Proof. Observe that

$$\begin{aligned} |\tilde{a}| &= \frac{1}{(\pi\hbar)^n} \left| \int_{\mathbb{R}^n} a(y,\eta) G(x-y,\xi-\eta) dy d\eta \right| \\ &\leq \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} C G(x-y,\xi-\eta) dy d\eta && (24) \\ &< \infty && (25) \end{aligned}$$

where the last line follows because G is integrable. Also note that the constant C from line (24) is the constant bounding a that exists since $a \in S^0(1)$.

Now let $\alpha \in \mathbb{N}^{2n}$. Then we have that

$$\begin{aligned} |\partial^\alpha \tilde{a}| &= \left| \frac{1}{(\pi\hbar)^n} \partial^\alpha (a * G) \right| \\ &= \frac{1}{(\pi\hbar)^n} |(\partial^\alpha a) * G| && (26) \\ &= \frac{1}{(\pi\hbar)^n} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\partial^\alpha a(y,\eta)) G(x-y,\xi-\eta) dy d\eta \right| \\ &\leq \frac{1}{(\pi\hbar)^n} C_\alpha \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G(x-y,\xi-\eta) dy d\eta && (27) \\ &< \infty \end{aligned}$$

where we may move the derivative inside on line (26) because $G \in L^1$ and $a \in C^\infty$. The constant C_α on line (27) are the constants bounding the derivative $\partial^\alpha a$ that exist because $a \in S^0(1)$. □

We may now prove the following proposition relating a and $\tilde{a}$.

Proposition 1. *For $a \in S^0(1)$ we have that $\tilde{a} - a = O(\hbar)$.*

Proof. We use a second order Taylor expansion about (x, ξ) to write the estimate:

$$
\begin{aligned}
&\tilde{a}(x,\xi) - a(x-y,\xi-\eta) = \\
&\qquad a(x,\xi) - \left[\sum \frac{\partial a}{\partial \xi_j}(x,\xi)\eta_j - \sum \frac{\partial a}{\partial x_j}(x,\xi)y_j\right] \\
&\qquad - \frac{1}{2}\sum\left[\frac{\partial^2 a}{\partial \xi_j \xi_k}(x,\xi)\eta_j\eta_k + 2\frac{\partial^2 a}{\partial \xi_j \partial x_k}(x,\xi)\eta_j y_k + \frac{\partial^2 a}{\partial x_j \partial x_k}(x,\xi)y_j y_k\right] \\
&\qquad - a(x,\xi)
\end{aligned}
$$

Now we note that, since $\int G = 1$,

$$
\begin{aligned}
|\tilde{a} - a| &= |a - a * G| \\
&= \left| a(x,\xi) - \int (a(x-y,\xi-\eta)G(y,\eta))dyd\eta \right| \\
&= \left| \int (a(x,\xi) - a(x-y,\xi-\eta)G(y,\eta))dyd\eta \right| \\
&\leq \int \left| \left[\sum \frac{\partial a}{\partial \xi_j}(x,\xi)\eta_j - \sum \frac{\partial a}{\partial x_j}(x,\xi)y_j\right]\right| G(y,\eta)dyd\eta \qquad (28)\\
&\quad + \frac{1}{2}\int \left|\sum\left[\frac{\partial^2 a}{\partial \xi_j \xi_k}(x,\xi)\eta_j\eta_k + 2\frac{\partial^2 a}{\partial \xi_j \partial x_k}(x,\xi)\eta_j y_k + \frac{\partial^2 a}{\partial x_j \partial x_k}(x,\xi)y_j y_k\right]\right| \\
&\qquad G(y,\eta)dyd\eta \qquad (29)
\end{aligned}
$$

We now note that the integrals on line (28) vanish because $\eta_j G(y,\eta)$ is an even function, and similarly for $y_j G(y,\eta)$. We note that we can bound the first term on line (29) by (note that the others can be bounded by the same argument)

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [C_1|\eta|^2 + 2C_2|\eta||y| + C_3|y|^2]G(y,\eta)dyd\eta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [C_1|\eta|^2 + 2C_2|\eta||y| + C_3|y|^2]e^{-(|y|^2+|\eta|^2)/\hbar}dyd\eta
\end{aligned}
$$

and now by a change of variables $y \mapsto \hbar^{1/2}y, \eta \mapsto \hbar^{1/2}\eta$ in the integral we get that $|\tilde{a} - a| = O(\hbar)$. $\square$

At this point we are ready to define our quantization.

Definition 4. *We define the quantization* Op^{aw} *by*

$$\mathrm{Op}^{aw}(a) = \mathrm{Op}^{w}(\tilde{a}).$$

where Op^{w} *is the Weyl quantization.*

We now observe that because of proposition 1 it follows from the Calderon-Vaillancourt theorem that in the limit $\hbar_0 \to 0$ we have

$$||\mathrm{Op}^{aw}(a) - a^w||_{L^2 \to L^2} = O(\hbar).$$

We now prove the key property of Op^{aw}.

Proposition 2. *Let* $a \in S^0(1)$ *and* $a \geq 0$*. Then* $\mathrm{Op}^{aw}(a) \geq 0$*.*

Proof. We observe that

$$\begin{aligned}
\mathrm{Op}^{aw}(a)u &= \tilde{a}^w u \\
&= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^{2n}} e^{i/\hbar\langle x-y,\xi\rangle} \tilde{a}\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi \\
&= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} e^{i/\hbar\langle x-y,\xi\rangle} \int_{\mathbb{R}^{2n}} a(r,s) e^{-(|(x+y)/2-r|^2+|\xi-s|^2)} dr ds u(y) dy d\xi
\end{aligned}$$

and hence we have

$$\langle Op^{aw}(a)u, u\rangle = C \int_{\mathbb{R}^{2n}} e^{i/\hbar\langle x-y,\xi\rangle} \int_{\mathbb{R}^{2n}} a(r,s) \\ e^{-(|(x+y)/2-r|^2+|\xi-s|^2)} dr ds u(y) dy d\xi \overline{u(x)} dx$$

Now notice that we have an integral function and its complex conjugate in the expression above (splitting the $e^{i/\hbar\langle x-y,\xi\rangle}$ factor) and we can write this integral in the form

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(r,s) |U\overline{U}(r,s)| e^{|\xi-s|^2} d\xi dr ds = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(r,s) |U(r,s)|^2 e^{|\xi-s|^2} d\xi dr ds$$

where U is a function of the form $U = \int e^{-i/\hbar\langle y,\xi\rangle} u(y) dy$.

Since everything in the above expression is non-negative, the claim follows. $\square$

We are now ready to prove the theorem. We know that since the quantization is asymptotically equivalent to the Weyl quantization, we write $\mathrm{Op}^{aw}(a) = a^w(x, \hbar D_x) + \hbar r^w(x, \hbar D_x)$ where $r^w \in S^0(1)$. Now since $\mathrm{Op}^{aw} \geq 0$ we know that

$$\langle [a^w(x, \hbar D_x) + \hbar r^w(x, \hbar D_x)]u, u\rangle_{L^2} \geq 0.$$

This is equivalence to writing

$$\langle a^w u, u\rangle \geq -\hbar \langle r^w u, u\rangle \geq -\hbar C \, ||u||^2 \, ,$$

where the last inequality follows from L^2 boundedness.

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