MATH 581 ASSIGNMENT 6

DUE FRIDAY MARCH 30

1. a) *Maxwell's equations* for 3 dimensional electromagnetism in vacuum are

$$\partial_t E = \nabla \times B, \qquad \partial_t B = -\nabla \times E, \tag{1}$$

and

$$\nabla \cdot E = 0, \qquad \nabla \cdot B = 0, \tag{2}$$

where $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ are the electric and magnetic fields, respectively. Show that the system (1) is symmetric hyperbolic. Then show that the constraints (2) are preserved by the evolution, i.e., that if one starts with initial data satisfying the constraints (2), and if E and B evolve according to (1), then (2) will be satisfied for all time.

b) Consider the second order system

$$\partial_t^2 u = \sum_{j,k=1}^{n} A_{j,k} \partial_j \partial_k u, \qquad u|_{t=0} = f, \qquad \partial_t u|_{t=0} = g,$$

which generalizes the wave equation. Here we suppose $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, and that $A_{j,k}$ are $m \times m$ matrices. Let us say that the system is *symmetric hyperbolic* if the matrices $A_{j,k}$ are symmetric and positive definite, which then we assume. Prove (either by reducing it to a first order system or directly) that the above Cauchy problem is well-posed in the following sense: For any initial data $(f, g) \in H^s \times H^{s-1}$ with some $s \in \mathbb{R}$, there exists a unique solution $u \in C^0(\mathbb{R}, H^s)$ with $\partial_t u \in C^0(\mathbb{R}, H^{s-1})$, satisfying

$$\|u(t)\|_{H^s} \leq C\|f\|_{H^s} + \alpha(t)\|g\|_{H^{s-1}}, \qquad \|\partial_t u(t)\|_{H^{s-1}} \leq C(\|f\|_{H^s} + \|g\|_{H^{s-1}}),$$

where C is a constant, and the function $\alpha(t)$ grows slower than some polynomial for large t.

For isotropic and homogeneous materials, the *elastodynamics equations* are given by

$$\partial_t^2 u = \mu \Delta u + \lambda \nabla(\nabla \cdot u),$$

where $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the displacement field, and μ and λ are real parameters. In components, it reads

$$\partial_t^2 u_k = \mu \Delta u_k + \lambda \partial_k (\partial_1 u_1 + \dots \partial_n u_n), \qquad k = 1, \dots, n.$$

Determine the values of the parameters μ and λ for which the system is symmetric hyperbolic.

Date: Winter 2012.

2. Consider the n-dimensional *Navier-Stokes equations*
$$\partial_t u = \Delta u - u \cdot \nabla u - \nabla p, \qquad \nabla \cdot u = 0, \tag{3}$$
where $u : \mathbb{R}^n \times \overline{\mathbb{R}}_+ \to \mathbb{R}^n$ is the velocity field, and $p : \mathbb{R}^n \times \overline{\mathbb{R}}_+ \to \mathbb{R}$ is the pressure field. For clarity, in components the first of the Navier-Stokes equations would read
$$\partial_t u_k + (u_1\partial_1 + \ldots + u_n\partial_n)u_k = \Delta u_k - \partial_k p, \qquad k = 1, \ldots, n.$$
Let us overload the notation $H^s(\mathbb{R}^n)$ so that it denotes also the space of vector fields with each component lying in $H^s(\mathbb{R}^n)$. Define the *Leray projector* $P : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ by
$$\widehat{Pu}(\xi) = \hat{u}(\xi) - (\hat{u}(\xi) \cdot \xi)\xi/|\xi|^2.$$
In components, it is
$$\widehat{(Pu)}_k(\xi) = \hat{u}_k(\xi) - \frac{\xi_1 \hat{u}_1(\xi) + \ldots + \xi_n \hat{u}_n(\xi)}{\xi_1^2 + \ldots + \xi_n^2}\xi_k = (\delta_{km} - \frac{\xi_k \xi_m}{|\xi|^2})\hat{u}_m(\xi),$$
where the summation convention is understood in the latter expression. In Fourier space, the divergence free condition $\nabla \cdot u = 0$ is simply $\xi \cdot \hat{u}(\xi) = 0$, so the Leray projector projects onto the divergence free space. A formal application of P to (3) gives
$$\partial_t u = \Delta u - P(u \cdot \nabla u). \tag{4}$$
   a) For which values of $s \in \mathbb{R}$ is $P : H^s \to H^s$ bounded?
   b) Show that in an appropriate sense, the two formulations (3) and (4) are equivalent.
   c) Prove the local well-posedness of (4) in H^s for $s > \frac{n}{2} + 1$.
   d) In order to update the above result to a global well-posedness result, what kind of bound would you need?
   e) *Bonus problem*: Prove the global well-posedness of the 2 dimensional Navier-Stokes equations in H^s for $s > 2$.
3. Let $\Omega \subset \mathbb{R}^n$ be an open set, let $k \geq 0$ be an integer, and let $1 \leq p \leq \infty$. Then the *Sobolev space* $W^{k,p}(\Omega)$ by definition consists of those $u \in \mathscr{D}'(\Omega)$ such that $\partial^\alpha u \in L^p(\Omega)$ for each α with $|\alpha| \leq k$. Equip it with the norm
$$\|u\|_{W^{k,p}(\Omega)} = N(\{\|\partial^\alpha u\|_{L^p(\Omega)} : |\alpha| \leq k\}),$$
where N is a norm on the finite dimensional space $\{\lambda_\alpha \in \mathbb{R} : |\alpha| \leq k\}$. Obviously, the topology of $W^{k,p}(\Omega)$ does not depend on the choice of N, so one can pick N at their convenience.
   a) Show that $W^{k,p}(\Omega)$ is a Banach space for any $k \geq 0$ and $1 \leq p \leq \infty$.
   b) Show that $\mathscr{D}(\mathbb{R}^n)$ is a dense subspace of $W^{k,p}(\mathbb{R}^n)$, for any $k \geq 0$ and $1 \leq p < \infty$.
4. Recall that the *Sobolev inequality*
$$\|u\|_{L^q} \leq C\|u\|_{W^{1,p}}, \qquad u \in \mathscr{D}(\mathbb{R}^n), \tag{5}$$
with some constant $C = C(p, q)$, is valid when $1 \leq p \leq q < \infty$, and $\frac{1}{p} \leq \frac{1}{q} + \frac{1}{n}$.
   a) By way of a counterexample, show that the inequality (5) fails whenever $q < p$.
   b) Show that (5) fails when $\frac{1}{p} > \frac{1}{q} + \frac{1}{n}$.
   c) Show that (5) fails for $p = n$ and $q = \infty$ when $n \geq 2$.

d) Derive sufficient conditions on the exponents p, q, k, m under which the inequality

$$\|u\|_{W^{m,q}} \leq C\|u\|_{W^{k,p}}, \qquad u \in \mathscr{D}(\mathbb{R}^n),$$

is valid.

5. a) Let $\Omega \subset \mathbb{R}^n$ be a domain and let χ be a smooth function satisfying $\chi \in W^{\ell,\infty}(\Omega)$ for all ℓ. Show that $u \mapsto \chi u : W^{k,p}(\Omega) \to W^{k,p}(\Omega)$ is bounded for $k \geq 0$ and $1 \leq p \leq \infty$.

b) Consider the differential operator

$$L = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha,$$

on some domain $\Omega \subset \mathbb{R}^n$, where the coefficients satisfy $a_\alpha \in C^\infty(\Omega) \cap W^{\ell,\infty}(\Omega)$ for all ℓ. Show that $L : W^{k+m,p}(\Omega) \to W^{k,p}(\Omega)$ is bounded for $k \geq 0$ and $1 \leq p \leq \infty$.

c) Let $\Omega, \Omega' \subset \mathbb{R}^n$ be two bounded domains, with a diffeomorphism $\phi : \Omega \to \Omega'$ that can be extended to a diffeomorphism from a neighbourhood of Ω to a neighbourhood of Ω'. Prove that the pullback $\phi^* : W^{k,p}(\Omega') \to W^{k,p}(\Omega)$ is a bounded linear operator for $k \geq 0$ and $1 \leq p \leq \infty$.