MATH 581 - Final Project
Mean Curvature Flows

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1 Introduction

The mean curvature flow is part of the bigger family of geometric flows, which are flows on a manifold associated with some geometrical quantity. This family includes some very famous examples such as the *Calabi flow* and the *Ricci flow*. Those two example are defined intrinsiquely, meaning the the definition of the flow is done through some measure on the manifold, and doesn't depend on its imbedding on a higher dimension space. But this is not the case of the mean curvature flow, which is defined extrinsinquely. This conveniently allows to see the mean curvature flow as a deformation of a manifold.

Mean curvature flows are often given as a model for soap films. Such objects are driven by surface tension, which directly depend on the curvature of the soap film. The analogy is quite strong and reflects many properties of the mean curvature flow. For example, it is well known that soap films tend to converge to a minimal surface, which are critical points of the mean curvature flow under reasonable constraints.

This paper is organized as follows. Section 2 will give the main definitions and concepts used throughout the paper. Then section 3 will treat the case of convex initial conditions, proving asymptotic convergence to a spherical singularity. Section 4 will explore the concept of weak solutions after a singularity has occured, and section 5 will conclude and give some possible future work.

2 Definitions, intuition and examples

Let $\{M_t\}_{t\in\mathbb{R}^+}$ be a family of hypersurfaces of codimension 1 in $\mathbb{R}^{n+1}$, with M_0 given. Let these surfaces be represented locally by a diffeomorphism

$$F : U \subset \mathbb{R}^n \to F(U) \subset \mathbb{R}^{n+1}. \tag{1}$$

We say that M_t is moving by *mean curvature flow* (MCF) if it satisfies the equation

$$\partial_t F(x,t) = \vec{H}(x,t) \qquad \forall x \in U \tag{2}$$

where $\vec{H}(x,t)$ is the inwards mean curvature vector of the surface at position x and time t. Since the surface in imbedded, there is no problem in defining $\vec{H}$ with the principal curvatures λ_i and the outwards unit normal vector ν by

$$\vec{H} = -\vec{\nu}\frac{1}{n}\sum_{i=1}^{n}\lambda_i. \tag{3}$$

We will sometimes refer to the mean curvature as a scalar by calling it $H := n|H|$. Before investigating the propertiess of this equation, let's look at some examples.

2.1 Shriking sphere

A very simple example is when M_0 is a sphere of radius R_0. Any point on M_0 can be described by a vector $\theta(\theta_1, ..., \theta_{n-1})$ and a radius $R(t)$. The initial normal vector at any point on M_0 is $\frac{x}{|x|} = \theta$, and this vector doesn't change over time. The equation (3) for curvature thus gives

$$H(x) = -\theta\frac{1}{n}\sum_{i=1}^{n}\frac{1}{R(t)} = -\theta\frac{1}{R(t)} \tag{4}$$

so that the MCF equation (2) becomes

$$\partial_t(\theta R(t)) = -\theta\frac{1}{R(t)} \tag{5}$$

$$\Rightarrow \partial_t R(t) = -\frac{1}{R(t)} \tag{6}$$

$$\Rightarrow R(t) = \sqrt{R_0^2 - 2t}. \tag{7}$$

This trivial example exhibits many important features of the MCF. First, we see that since the radius of the sphere decreases, the area of the surface also decreases. We will see that this is a comon features of surfaces advected by MCF. Also, we remark that after a time $T = \frac{R_0^2}{2} < \infty$ the surface degenerates to a single point. It is important to notice that the speed at which the surface approaches the singularity is asymptotically infinite, which is also the case for many initial surfaces.

The mean curvature flow equation is parabolic, and thus has some of the smoothing properties of a heat equation for short times. But as we see in the case of the sphere, a singularity is created after a certain finite time. It is a comon feature of this equation to devellop such singularities, as will be pointed out for instance in section 3.

2.2 Self-shrinking surfaces

MCF being a parabolic equation, we see that it is invariant under the parabolic scaling

$$x \mapsto \lambda x \qquad t \mapsto \lambda^2 t \tag{8}$$

so we look for solutions of the form

$$M_t = \sqrt{-t} M^\star \qquad t < 0. \tag{9}$$

for some fixed surface $M^\star$. We mean here that the surface is shrinking by homothety from a negative time $-T$ up to $T = 0$ where the solution becomes singular. We call such surfaces *self-shrinking surfaces*. An example is the sphere described above, but there are other examples of self-shrinking surfaces. One obvious solution is the cylinder in $\mathbb{R}^3$, and another consist of a well chosen torus. But finding more example happens to be a difficult problem.

One great interest of such self-shrinking surfaces is that when singularities are created in MCF, the surface around them locally tends asymptotically to a self-shrinking surface. For instance, we will see in section 4 that any convex initial surface M_0 will converge to a point asymptotically like a sphere.

The cylinder is also a very important example because a surface with handles will tend to approach instabilities in a cylindrical manner. For instance, if we look at the torus in figure 1, wee se that since the curvature of the hole is greater than the curvature of the tube, a singularity will develop in the center and the surface will asymptotically shrink like a cylinder in the center until it reaches the singularity.

2.3 Normalized equation

We made the remark in the begining of this paper that MCF models the behavior of soap films. But we also showed in section 2.1 that a sphere will shrink to a point in a finite time, which is definitely not what happens to a spherical soap bubble. The problem is that such a bubble has the additional constraint of having to enclosed a constant volume of air, which was not imposed in our MCF model.

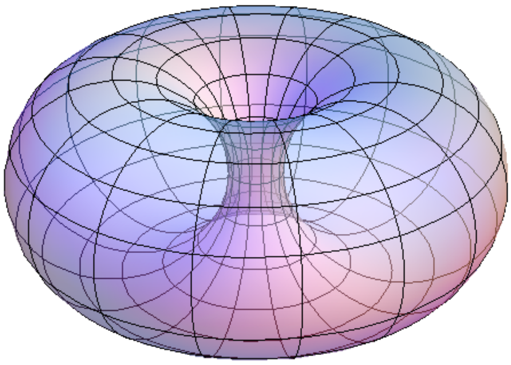


Figure 1: A torus that will develop a cylindrical singular point at its center.

But instead of dealing with constant enclosed volume, it is more convenient to deal with flows conserving the total area. This is easily obtained by defining the new surfaces $\tilde{M}_t$ by the diffeomorphisms

$$\tilde{F}(x,t) = \psi(t) \cdot F(x,t) \tag{10}$$

where $\psi(t)$ is the unique function so that the total area of M_t is conserved, i.e.,

$$\int_{\tilde{M}_t} d\tilde{\mu} = \int_{M_0} d\tilde{\mu}. \tag{11}$$

We also change the time variable in order to have more regularized modification of the equation. In other words, we want to avoid the speed of the manifold to go to infinity such as in the example of the sphere. We take the new time $\tilde{t}$ to be

$$\tilde{t}(t) = \int_0^t \psi^2(y)dy \tag{12}$$

so that the time increases following the modification in the total area of the surfaces. This gives the *normalized equation*

$$\partial_{\tilde{t}}\tilde{F}(x,\tilde{t}) = \tilde{H}(x) + \frac{1}{n}\frac{\int_{\tilde{M}_{\tilde{t}}} \tilde{H}^2 d\tilde{\mu}}{\int_{\tilde{M}_{\tilde{t}}} d\tilde{\mu}}\tilde{F}(x,\tilde{t}) \qquad \forall x \in \tilde{M}_{\tilde{t}} \quad \forall \tilde{t} \in \mathbb{R}^+ \tag{13}$$

with $\tilde{M}_0 = M_0$.

The space variable can also be translated so that the surfaces $\tilde{M}_t$ stay inside a ball of a given finite radius around the origin. This can be done, for instance, by recentering $\tilde{M}_t$ at everystep so that its center of mass is at the origin. Under these normalized equations, any convex initial surface will converge to a sphere of the same total area. Even though these equations have a more regular behavior and do not generate singularities as easily as the original MCF equation (2), they are more complicated to deal with. For this reason, we will still use (2) as our governing equation. Nevertheless, the normalized and original MCF equations are strongly related up until the time where singularities are created in the original MCF equation, so any results we will prove for equation (2) can be easily transposed to apply for equation (13).

3 Convex initial conditions

In this section, we explore in more details the case where M_0 is a convex surface governed by the MCF equation (2). We want to show the following three interesting results

Theorem 1. *Let $n \geq 2$ and M_0 be a uniformly convex initial surface, i.e., its principal curvatures are all strictly positive everywhere. Then*

- *i The evolution equation (2) with smooth initial conditions has a smooth solution on a finite time interval $0 \leq t \leq T < \infty$.*
- *ii The surfaces M_t converge to a single point asymptotically spherically as $t \to T$.*
- *iii The convergence rate is infinite as $t \to T$.*

Proof. We follow here the main ideas of the proof given by Huisken in [2]. The proof uses a lot of lemmas that are mostly computational, which we will use without proof.

First of all, since equation (2) is strongly parabolic, it has a solution on some interval $0 \leq t < T$. But we can also show that $T < \infty$. Let A denote the second fundamental form of the surface. We have from direct calculations the following evolution equation for the mean curvature H :

$$\partial_t H = \Delta H + H|A|^2 \geq \Delta H + \frac{1}{n}H^3 \tag{14}$$

which possesses a maximum principle. Then take ϕ to be the solution of the ODE

$$\partial_t \phi = \frac{1}{n}\phi^3 \qquad \phi(0) = H_{\min}(0) \tag{15}$$

where $H_{\min}(0)$ is the minimal mean curvature of the initial condition, which is positive since we suppose that the initial surface is convex. This implies

$$\phi(t) = \frac{H_{\min}(0)}{\sqrt{1 - \frac{2t}{n}H^2_{\min}(0)}}. \tag{16}$$

Now we can take the equation (14) but apply it to $H - \phi$ (seeing ϕ as a function of t and x but constant in x), which gives

$$\partial_t(H - \phi) \geq \Delta(H - \phi) + \frac{1}{n}(H - \phi)^3 \tag{17}$$

and since the minimum of $(H - \phi)$ is zero by construction, the maximum principle gives

$$H \geq \phi \tag{18}$$

on the interval $0 \leq t < T$ where the solution exists. But we have that $\phi \to \infty$ as $t \to \frac{n}{2H^2_{\min}(0)}$, and so we also have that $H \to \infty$ as $t \to \frac{n}{2H^2_{\min}(0)} < \infty$, so the solution will blow up in a finite time, which shows $T < \infty$ and concludes part $i)$ of theorem 1.

For part $ii)$, the idea is to show that the eigenvalues of the second fundamental from approach each other, and thus the surface approaches a sphere (this implication isn't trivial, but requires a lot of machinery that we will omit here. See [2] for a detailed proof). Denoting by κ_i the eigenvalues of A, we can use the identity

$$|A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}^{n}(k_i - k_j)^2 \tag{19}$$

so that $|A|^2 - \frac{1}{n}H^2$ can be used as a measure of the relative distances between the eigenvalues. We then want to show that this quantity becomes small, and it turns out that it is sufficient to show that it becomes small compared to H^2, i.e.,

$$|A|^2 - \frac{1}{n}H^2 \leq CH^{2-\delta} \qquad \text{for some } \delta > 0,\ C < \infty \tag{20}$$

to have that the convergence in asymptotically spherical. We first need some intermediate results, which we will use without proof. The proofs are mainly computational and do not give a lot of insight, but they can be found in [2]. We define $f_\sigma := \frac{|A|^2 - \frac{H^2}{n}}{H^{2-\sigma}}$.

Lemma 1. *Let $\alpha = 2 - \sigma$, then for any σ we have*

$$\partial_t f_\sigma \leq \Delta f_\sigma + \frac{2(\alpha-1)}{H}\langle \nabla_i H, \nabla_i f_\sigma \rangle - \frac{\epsilon^2}{H^\alpha}|\nabla H|^2 + \sigma |A|^2 f_\sigma \tag{21}$$

for $0 \leq t < T$ where $\langle \cdot, \cdot \rangle$ denotes the inner product on M_t

Lemma 2. *There exists a constant $C_1 < \infty$ depending only on M_0 such that for all p and σ satisfying*

$$p \geq \frac{100}{\epsilon^2} \qquad \text{and} \qquad \sigma \leq \frac{n\epsilon^3}{8\sqrt{p}} \tag{22}$$

the inequality

$$\left(\int_{M_t} f_\sigma^p d\mu\right)^{\frac{1}{p}} \leq C_1 \tag{23}$$

holds for $0 \geq t < T$.

Lemma 3. *Suppose that*

$$p \geq \left(\frac{16m}{n\epsilon^3}\right)^2 \qquad \text{and} \qquad \sigma \leq \frac{n\epsilon^3}{16\sqrt{p}} \tag{24}$$

then we have

$$\left(\int_{M_t} H^m f_\sigma^p d\mu\right)^{\frac{1}{p}} \leq C_1 \tag{25}$$

for $0 \leq t < T$.

Lemma 4. *For all Lipschitz functions v on M we have*

$$\left(\int_M |v|^{\frac{n}{n-1}} d\mu\right)^{\frac{n-1}{n}} \le c(n)\left(\int_M |\nabla v| d\mu + \int_M H|v| d\mu\right) \tag{26}$$

Now equipped with these lemmas, we can attack the proof of part *ii*) of theorem 1.

Define $A(k)$ to be the set over which $f_\sigma > k$ and define also $f_{\sigma,k} := \max(f_\sigma - k, 0)$. From lemma 2 we derive that

$$\partial_t \int_{A(k)} f^p_{\sigma,k} d\mu + \frac{1}{2}p(p-1)\int_{A(k)} |\nabla f_\sigma|^2 f^{p-2}_{\sigma,k} d\mu \le \sigma p \int_{A(k)} H^2 f^{p-1}_{\sigma,k} f_\sigma d\mu. \tag{27}$$

Also, we have on $A(k)$ that

$$\frac{1}{2}p(p-1) f^{p-2}_{\sigma,k} |\nabla f_\sigma|^2 \ge |\nabla f^{p/2}_{\sigma,k}|^2 \tag{28}$$

and thus by taking $v = f^{p/2}_{\sigma,k}$ we have

$$\partial_t \int_{A(k)} v^2 d\mu + \int_{A(k)} |\nabla v|^2 d\mu \le \sigma p \int_{A(k)} H^2 f^p_\sigma d\mu. \tag{29}$$

Now using lemma 4 we have

$$\left(\int_M v^{2q} d\mu\right)^{\frac{1}{q}} \le c(n)\int_M |\nabla v|^2 d\mu + c(n)\left(\int_{\mathrm{supp}(v)} H^n d\mu\right)^{\frac{2}{n}} \left(\int_M v^{2q}\right)^{\frac{1}{q}} \tag{30}$$

with some constant $c(n)$ depending on n and $q = \frac{n}{n-2}$ if $n > 2$ and $q = \infty$ if $n = 2$. Since we have that v is supported on a subset of $A(k)$ by construction, we can use lemma 3 with $p \ge \frac{2^8}{\epsilon^6}$ and $\sigma \le \frac{n\epsilon^3}{16\sqrt{p}}$ to get

$$\left(\int_{\mathrm{supp(v)}} H^n d\mu\right)^{\frac{2}{n}} \le k^{-2p/n}\left(\int_{A(k)} H^n f^p_\sigma d\mu\right)^{\frac{2}{n}} \le k^{-2p/n} C_1^{2p/n}. \tag{31}$$

We thus have, for large enough k, say $k \ge k_1$, that

$$\sup_{[0,T]} \int_{A(k)} v^2 d\mu + c(n)\int_0^T \left(\int_{A(k)} v^{2q} d\mu\right)^{\frac{1}{q}} dt \le \sigma p \int_0^T \int_{A(k)} H^2 f^p_\sigma \, d\mu \, dt. \tag{32}$$

Also, we directly have from L^p spaces inequalities that

$$\left(\int_{A(k)} v^{2\left(\frac{1}{a/q+(1-a)}\right)} d\mu\right)^{\frac{a}{q}+(1-a)} \le \left(\int_{A(k)} v^{2q} d\mu\right)^{a/q} \left(\int_{A(k)} v^2 d\mu\right)^(1-a) \tag{33}$$

which leads to

$$\left(\int_0^T\int_{A(k)} v^{2\left(\frac{1}{a/q+(1-a)}\right)}\,d\mu\,dt\right)^{\frac{a}{q}+(1-a)} \leq c(n)\sigma p||A(k)||^{1-\frac{1}{r}}\left(\int_0^T\int_{A(k)} H^{2r}f_\sigma^{pr}\,d\mu\,dt\right)^{\frac{1}{r}} \tag{34}$$

if we pick $r>1$ and where $||A(k)|| = \int_0^T\int_{A(k)} d\mu\,dt$. Then, if we choose

$$p \geq \frac{r2^{10}}{\epsilon^6} \qquad \text{and} \qquad \sigma \leq \frac{\epsilon^6}{2^9\sqrt{r}} \tag{35}$$

we get by lemma 3 and Holder inequalities that

$$f_\sigma \leq k_1 + \left(C_2 2^{p\gamma/(\gamma+1)}||A(k_1)||^{\gamma-1}\right)^{\frac{1}{p}} \tag{36}$$

with $\gamma = 2-\frac{1}{q}-(1-a)-\frac{1}{r}$. We can show that the total area of the surfaces M_t is decreasing over time (see [2]) and we already showed that T in bounded, so $||A(k)||$ is bounded, and we get as wanted that

$$f_\sigma \leq C \tag{37}$$

$$\Rightarrow |A|^2 - \frac{1}{n}H^2 \leq CH^{2-\delta} \tag{38}$$

which proves part *ii*) of theorem 1.

Now for part *iii*), we want to show that the speed at which the surface collapses to a point is infinite. We need the three following lemmas from [2]. We omit the proofs since they are mainly computational and not very insightful.

Lemma 5. *Let g_{ij} be a time dependant metric on a compact manifold M for $0\leq t< T<\infty$. Suppose*

$$\int_0^T \max_M |\partial_t g_{ij}|\,dt \leq C < \infty \tag{39}$$

where

$$|\partial_t g_{ij}| = \sqrt{g_{ik}g_{jl}\left(\partial_t g_{ij}\right)\left(\partial_t g_{kl}\right)} \tag{40}$$

with the convention of summation over repeated indices. Then the metrics $g_{ij}(t)$ for all different times are equivalent (i.e. they generate the same topology) and they converge uniformly as $t\to T$ to a positive definite metric tensor $g_{ij}(T)$ which is continuous and also equivalent to the $g_{ij}(t)$.

Lemma 6. *The metrics g_{ij} satisfy the equation*

$$\partial_t g_{ij} = -2Ha_{ij} \tag{41}$$

where the a_{ij} are the elements of the second fundamental form A.

Lemma 7. *If the second fundamental form A satisfies*

$$\max_{M_t} |A|^2 \leq C \qquad \textit{on } 0 \leq t < T < \infty \tag{42}$$

then

$$|\nabla^m A| \leq C_m \qquad \forall m. \tag{43}$$

We then proceed by contradiction. Suppose that

$$\max_{M_t} |A|^2 \leq C \qquad \text{on } 0 \leq t < T \tag{44}$$

where we know that $T < \infty$ from part i) of this theorem. From the MCF equation (2) we have directly that

$$|F(x,a) - F(x,b)| \leq \int_a^b H(x,t)dt \tag{45}$$

for $0 \leq a \leq b < T$. Since we suppose that $|A|$ is bounded, H is bounded, and F converges to a continuous function $F(x,T)$ as $t \to T$. But we need to ensure that this map still represents a limit surface M_T. But we have, using together the fact that all the M_t are diffeomorphic by constructrion, lemma 6, our assumption, and the fact that $T < \infty$, that

$$\int_0^T \max_{M_t} |\partial_t g_{ij}| dt \lesssim \int_0^T \max_{\tau} \max_{M_\tau} |\partial_t g_{ij}| dt \tag{46}$$

$$\leq \int_0^T \max_{M_0} |2H|A|\,| dt \tag{47}$$

$$\leq 2C^2 \int_0^T dt \tag{48}$$

$$\leq C_1 < \infty \tag{49}$$

which implies by lemma 5 that the metrics converge to something continuous as $t \to T$. Lemma 7 then tells us that the surface actually converge to a smooth surface M_T. But then the solution can be continued beyond T, which contradicts the fact that $T < \infty$, and thus proves that the curvature is unbounded when the surface collapses to a singular point, which also means that the convergence speed is infinite. This concludes the proof of part iii), and concludes also the proof of theorem 1. $\square$

The results of theorem 1 also give intuitive information about the normalized equations (13). First, since the convergence is spherical, we can show that this automatically causes the normalized surfaces $\tilde{M}_t$ to converge to a sphere. Also, since the convergence rate as $t \to T$ goes to infinity, the interval $0 \leq t \leq T$ maps to $0 \leq \tilde{t} < \infty$, giving a smooth solution for the normalized surfaces for any positive time $\tilde{t}$. These results about the normalized equations are proven rigourously in sections 9 and 10 of [2].

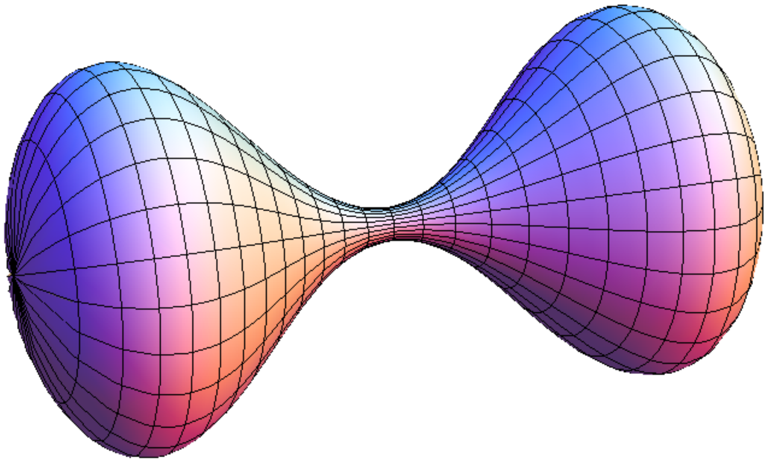


Figure 2: A barbel with a very thin neck that will create a singularity and transfom into two blobs under MCF.

4 Solutions beyond singularities

We showed in the previous section that convex initial surfaces will collapse to a point in a finite time T under mean curvature flow. In that case, there is thus no need to describe what happens after the singularity is created. But for other cases, the solution may still make sense after the first singularity is created. For instance, if we look at the example of the torus from section 2.2, the hole in the torus will collapse to a point in a finite time T_1, and the outer part of the surface will remain and will continue to evolve up until it collapses to a set of measure 0 in a larger but still finite time T_2. Another example given by Grayson [4] is the barbel in figure 2. The curvature in the very thin neck in the middle of this surface will cause the middle region to collapse to a point, separating the initial shape into two surfaces that will both collapse to a point. The problem is that when transitioning from $T_1 - \epsilon$ to $T_1 + \epsilon$, the diffeomorphism approach we have been using so far fails to describe what happens is a continuous manner.

For the MCF, the classification of the different types of possible singularities is not complete yet, and it is thus impossible to give a general solution by listing all the possible cases. We therefore use another method to describe the surfaces, which will give a suitable definition of a weak solution after singularities occur.

4.1 Level set functions

Any imbedded manifold M of codimension 1 in $\mathbb{R}^n$ can be represented as the zero level set of a function, i.e.,

$$\exists u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \qquad \text{such that} \qquad M_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}. \tag{50}$$

Such a function u is called a *level set function* for M. Of course, the choice of u is not unique, but the choice of level set function for the initial surface is of small importance for what follows. Still, we can choose u to be at least continuous and we use the convention that u is positive inside the surface and negative outside of it. The goal is then to transpose the MCF equation for M_t into an equation for u, so that the zero level of u still represents M_t for any positive time. Once again, there are a lot of evolution equations for u that would have this property, but if we impose that the equations evolves all of the level sets of u by MCF, then the choice of equation is unique (see [5]) and given by

$$\partial_t u - |\nabla u| \, \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = 0 \tag{51}$$

where ∇u represents the spatial gradient. This equation is called the *level set equation for MCF.*

The level set function representation has the useful property of being able to describe many geometrical properties of M in a simple way. For instance, the normal velocity V is given by $\frac{\partial_t u}{|\nabla u|}$, the inward pointing unit normal vector of M at any point is given by $\frac{\nabla u}{|\nabla u|}$ and the curvature is therefore given by $\nabla \cdot \frac{\nabla u}{|\nabla u|}$. The computations to arrive at these quantities are rather easy. For instance, for the normal vector, take a curve $\gamma(s)$ on M for a fixed time. Let $x_0 := \gamma(0)$. Then we have

$$u(\gamma(s)) = 0 \tag{52}$$

and differentiating by s we get

$$\nabla u(\gamma(0)) \cdot \partial_s \gamma(0) = 0 \tag{53}$$

and this is true for any curve going through x_0 at $s = 0$, so ∇u is perpendicular to any tangeant vector to the curve, so ∇u is a normal vector. We then get a unit vector by normalizing it, which gives $\frac{\nabla u}{|\nabla(u)|}$.

The derivation of the level set equation is also direct. We have from standard differential geometry that

$$H = \nabla \cdot \vec{n} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) \tag{54}$$

and since the MCF equation can be written as

$$V = H \tag{55}$$

we get directly

$$\frac{\partial_t u}{|\nabla u|} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) \tag{56}$$

or

$$\partial_t u - |\nabla u| \, \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = 0 \tag{57}$$

which is equation (51).

As shown in the 2D toy example of figure 3, the advantage of the level set approach is that a singular M can still be obtained from a smooth u. Also, it can be shown that the solution of (51) exists and is unique for a given initial level set function (see section 4.2). Furthermore, the zero level set of u coincides with M_t as defined in section 2 for times $0 \leq t \leq T$ where the solution by the diffeomorphism approach exists, so this level set approach is really an extension of the concept of classical solutions for mean curvature flows.

4.2 Perron's method

We show here a quick overview of how to show existence of solutions to the level set equation (51) by Perron's method. The details can be found in chapter 2 of [3].

The equation we are trying to solve is parabolic, but very degenerate, and we will use the concepts of viscosity solutions to get more regular properties on the equation by treating it as a degenerate elliptic equation.

To simplify the notation, we will use a more generic notation. We will look at the equation

$$\partial_t u(z) + F(z, u(z), \nabla u(z), \nabla^2 u(z)) = 0 \tag{58}$$

for which the MCF is a particular case by taking

$$F(a, b, c, d) = -\text{trace} \left(I - \frac{c \otimes c}{|c|^2} \right) d. \tag{59}$$

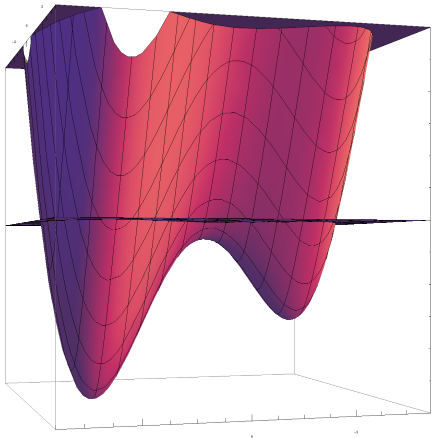
We have indeed that in that case, F is degenerate elliptic, i.e.,

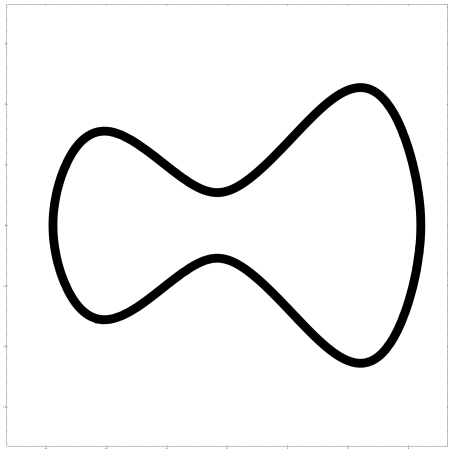
$$F(a, b, c, d_1) \leq F(a, b, c, d_2) \qquad \text{for } d_1 \geq d_2. \tag{60}$$

We will want to look at the solutions on a domain $\Theta = \Omega \times [0, T)$. This is not a problem for level set function, since we can take Ω large enough to contain all zero level sets, and extend it by a constant outside Ω. We define the upper and lower semicontinuous envelopes $u^\star$ and $u_\star$ by

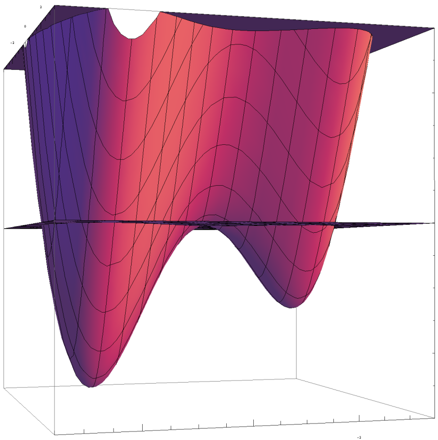
$$u^\star(z) = \lim_{r \searrow 0} \sup\{u(w) \mid w \in B_r(z)\} \tag{61}$$

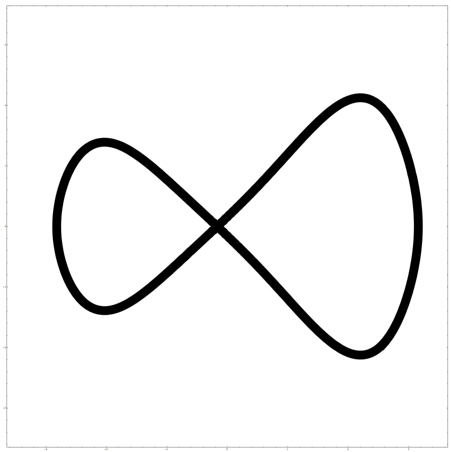
$$u_\star(z) = \lim_{r \searrow 0} \inf\{u(w) \mid w \in B_r(z)\} \tag{62}$$



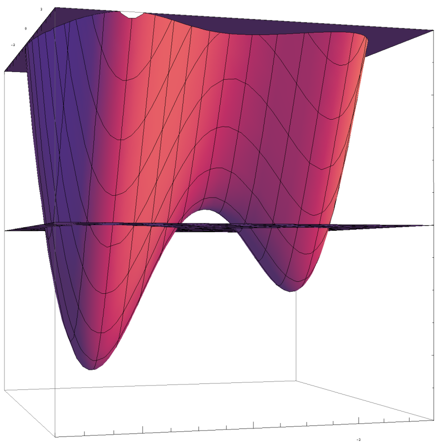


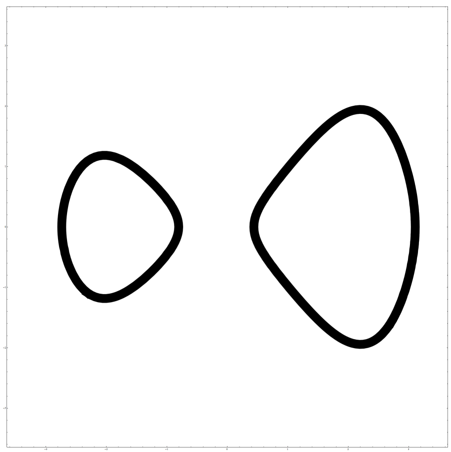
(a) Level set function at t=0. (b) M at t=0.





(c) Level set function at t=3. (d) M at t=3.





(e) Level set function at t=6. (f) M at t=6.

Figure 3: Toy example of a level set function. We use the initial level set function $u_0(x) = (x^4 - 9x^2 - 3x) + 20 \sqrt[1+y^2]{-}23$ and suppose that the solution to the level set MCF equation is $u(x,t) = u_0(x) + t$. We show here the solution for times 0, 3 and 6. On the left is the level set function along with the zero plane, and on the left is the zero level set. We see that the level set function has no problems when dealing with the singularity of M at $t = 3$.

which makes the upper and lower parts of u continuous on open sets. The whole idea of Perron's method is based on sub- and supersolutions. A function u is called a sub- (resp. super-) solution if

$$\partial_t u + F(z, u(z), \nabla u(z), \nabla^2 u(z)) \leq 0 \qquad (resp. \geq 0) \tag{63}$$

on Θ.

With these definitions, we can state the fundamental principle of Perron's method.

Theorem 2. (Comparison principle.) *If u and v are respectively sub- and supersolutions of* (66) *in Θ, then if $u^\star \leq v_\star$ on the parabolic boundary of Θ*

$$\partial_\sqcup \Theta := \Omega \times 0 \cup \partial\Omega \times [0, T) \tag{64}$$

then $u^\star \leq v_\star$ also everywhere in Θ.

This result is used to create a solution by squeezing it between a sub- and a supersolution, as we will see shortly. We need two lemmas before concluding this section.

Lemma 8. *Let S be a set of subsolutions of* (66)*. Then the function u defined by*

$$u(z) = \sup\{v(z)|v \in S\} \tag{65}$$

is also a subsolution.

Lemma 9. *Let h be a supersolution and let S_h be the set of all subsolutions v of* (66) *that satisfy $v \leq h$ everywhere in Θ. Then if a particular $v_0 \in S_h$ is not a supersolution, then there exists a function $w \in S_h$ such that $v_0(z) < w(z)$ for some point $z \in \Theta$.*

We can now state the existence result for the level set equation.

Theorem 3. *Let h_- and h_+ be sub- and supersolution, and suppose $h_- \leq h_+$. Suppose also that $h_{-\star} > -\infty$ and $h_+^\star < \infty$. Then there exists a solution of the level set equation* (66) *satisfying $h_- \leq u \leq h_+$.*

Proof. The proof easily follows from the previous lemmas. Since $h_- \leq h_+$, the set S_{h_+} as defined in lemma 9 is not empty. Therefore, we can create u as in lemma 8 by $u(z) = \sup\{v(z)|v \in S_{h_+}\}$. This gives a well defined function since $-\infty < u < \infty$. The same lemma thus gives that u is a subsolution. u is also a supersolution, because if it was not, then by lemma 8 there would exist a member w of S_{h_+} so that $u \not> w$, which is impossible by definition of u. Being a sub- and supersolution, u is therefore a solution of the level set equation. The last part of the theorem follows by definition of u. □

The problem of existence is therefore translated into finding sub- and supersolutions of (66). But this is easier than finding actual solutions, and we can even use sequences of solutions to define those sub- and supersolutions, meaning that if u_ϵ is a sub- or supersolution of

$$\partial_t u_\epsilon(z) + F_\epsilon(z, u(z), \nabla u(z), \nabla^2 u(z)) = 0 \tag{66}$$

for a sequence $F_\epsilon \to F$ as $\epsilon \searrow 0$, and if $u_\epsilon \to u$ uniformly for some u, then this u is a sub- of supersolution of (66).

If one is able to find sub- and supersolutions u_- and u_+ such that $u_+^\star = u = u_{-\star}$, then by Perron's method u is automatically a solution. Moreover, by looking for u_- and u_+ that are equal on $\partial_\sqcup\Omega$, we can apply the comparison principle to get $u^\star \leq u_\star$ everywhere in Θ, so u is continuous.

The comparison principle also gives us uniqueness of the solution. If u and v are solutions with same value on $\partial_\sqcup\Omega$, then we have $u \leq v$ and $v \leq u$ on the boundary so by the comparison principle we have $u \leq v$ and $v \leq u$ everywhere in Θ, so $u = v$.

5 Conclusion

We explored in this paper some of the main characteristics of the mean curvature flow. We spent a lot of time showing properties in the case where the initial surface is convex, concluding that it will develop a singulaity in a finite time. We then explored a possible option to extend the solutions beyond the formation of singularities in the sense of weak solutions, focusing mainly on the level set approach.

Most of the work we did here can be transposed without too much difficulty to other geometric flows. Using the notation of section 4.2, if we can write another geometric flow in the form

$$\partial_t u(z) + F(z, u(z), \nabla u(z), \nabla^2 u(z)) = 0 \tag{67}$$

with F having properties similar to the MCF, the arguments are similar. The book by Giga [3] treats generic surface evolutions equations this way. To give an example, to apply Perron's method to other geometric flows, having that F is degenerate elliptic is sufficient to use the same ideas.

Of course, many other properties of the level set approach we used to extend the classical solutions beyond singularities would need to be explored. Giga [3] does a good job of rigourously showing convergence of the viscosity solutions in very general cases, covering for instance higher dimension equations and some boundary problems. It also shows another approach to weak solutions using a set-theoritic technique.

The problems related to the mean curvature flows are in general very hard. For instance, the types of singularities that can be created from smooth initial surfaces are hard to classify, and numerical tools are often needed in order to find new results. The equation is nevertheless very useful a wide variety of domains, so every progress done has a lot of impact.

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