Lecture[1] 7

Restriction and Support

Definition 1. *Let $u \in \mathscr{D}'(\Omega)$ and consider $\omega \subset \Omega$ open. The **restriction** $u|_\omega \in \mathscr{D}'(\omega)$ of u to ω is defined by:*

$$< u\big|_{\omega}, \phi > = < u, \phi >, \qquad \forall \phi \in \mathscr{D}(\omega).$$

Theorem 1. *Let $u \in \mathscr{D}'(\Omega)$. The following hold:*

a) $u\big|_{\Omega} = u$.

b) If $\sigma \subseteq \omega \subseteq \Omega$ is open, then $\left(u\big|_{\omega}\right)\big|_{\sigma} = u\big|_{\sigma}$.

c) Let $\{\omega_\alpha\}$ be an open cover of Ω, then $u\big|_{\omega_\alpha} = 0 \implies u = 0$.

d) Let $\{\omega_\alpha\}$ be an open cover. Suppose $u_\alpha \in \mathscr{D}'(\omega_\alpha)$ satisfying

$$u_\alpha\big|_{\omega_\alpha \cap \omega_\beta} = u_\beta\big|_{\omega_\alpha \cap \omega_\beta}, \forall \alpha, \beta \implies \exists u \in \mathscr{D}'(\Omega) \ s.t \ u\big|_{\omega_\alpha} = u_\alpha, \ \forall \alpha.$$

Proof. $d)$ Let $\phi \in \mathscr{D}(\Omega)$, $K = supp\phi$ compact and consider a partition of unity $\{\chi_\alpha\}$ of K subordinate to $\{\omega_\alpha\}$.

$$< u, \phi > := \sum_\alpha < u_\alpha, \chi_\alpha \phi > .$$

Suppose $\{\xi_\beta\}$ another partition of unity, we have

$$\sum_\beta < u_\beta, \xi_\beta \phi > = \sum_\beta < u_\beta, \sum_\alpha \chi_\alpha \xi_\beta \phi > \tag{1}$$

$$= \sum_\alpha \sum_\beta < u_\beta, \chi_\alpha \xi_\beta \phi > \tag{2}$$

$$= \sum_\alpha \sum_\beta < u_\alpha, \chi_\alpha \xi_\beta \phi > \tag{3}$$

$$= \sum_\alpha < u_\alpha, \sum_\beta \xi_\beta \chi_\alpha \phi > \tag{4}$$

$$= \sum_\alpha < u_\alpha, \chi_\alpha \phi > \tag{5}$$

To establish continuity, consider $K \subset \Omega$ compact, $\phi \in \mathscr{D}(K)$,

$$| < u, \phi > | \leq \sum_\alpha | < u_\alpha, \chi_\alpha \phi > | \tag{6}$$

$$\leq \sum_\alpha C_\alpha \|\chi_\alpha \phi\|_{C^{m_\alpha}(K)} \tag{7}$$

$$\leq \sum_\alpha C'_\alpha \|\phi\|_{C^{m_\alpha}(K)} \tag{8}$$

$$\leq C\|\phi\|_{C^m(K)}. \tag{9}$$

□

[1] Notes by Ibrahim Al Balushi

Definition 2. *Let $u \in \mathscr{D}'(\Omega)$. The* ***support*** *of u,*

$$supp\ u = \Omega \backslash \bigcup \{\omega \subset \Omega\ open :\ u\big|_{\omega} = 0\}$$