Lecture¹ 6

Last time

$$T_u(\phi) = \int_{\Omega} u\phi, \quad j: u \mapsto T_u: C(\Omega) \to RM(\Omega).$$

For K compact supporting $\phi, \phi \in C_c^0(\Omega)$, the following estimate is used to prove continuity:

 $|T_u(\phi)| \le ||u||_{L^1(K)} \cdot ||\phi||_{C^0(K)} \le |K|||u||_{C^0(K)} ||\phi||_{C^0(K)}$

- $T_u: C_c^0(\Omega) \to \mathbb{R}$ continuous, $T_u \in RM(\Omega)$.
- $j: C(\Omega) \to RM(\Omega)$ continuous.
- j injective, $j : C(\Omega) \hookrightarrow RM(\Omega)$ continuous injection.
- $C(\Omega) \hookrightarrow L^1_{loc}(\Omega)$, injection.
- $L_{loc}^p \hookrightarrow L_{loc}^1(\Omega)$, for all $p \in [0, \infty]$.
- j extended to $j: L^1_{loc}(\Omega) \hookrightarrow RM(\Omega)$, continuous injection.

Injection of Duals

Definition 1. Let X, Y be topological vector spaces, $A : X \to Y$ linear. We define the adjoint $A' : X \to Y'$ for $y \in Y'$:

$$(A'y')(x) = y'(Ax), \quad \forall x \in X.$$

The following may be written notationally as $\langle A'y', x \rangle = \langle y', Ax \rangle$, $A'y' = y' \circ A$, where A'y' is the pullback of y' under $A = A^*y'$.

Equip X', Y' with their weak* topologies, if A continuous, then

 $| < A'y', x > | = | < y', Ax > | \implies A' : X' \to X' \ continuous.$

Lemma 1. $A: X \to Y$ continuous and linear A(X) dense in $Y \implies A'$ is injective.

$$Proof. \underbrace{(A'y')(x)}_{y'(Ax)} = 0 \text{ for all } x \in X \implies y' : \underbrace{Ax}_{z} \mapsto y'(Ax), \ z \in A(X).$$

Corollary 1. X, Y topological vector space. $j : X \hookrightarrow Y$ continuous injection with a dense image. Then,

 $j': Y' \to X'$

is continuous injection with dense image.

The elements of \mathcal{D}'^m are called *distributions of order* $\leq m$, cf. Figure 1, and the set of all distributions of finite order is denoted by $\mathcal{D}'_F = \bigcup_{m>0} \mathcal{D}'^m$.

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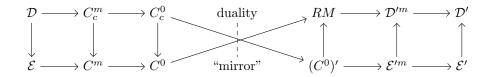


Figure 1: Each arrow represents a dense embedding. The domain Ω in each space is understood, e.g., $\mathcal{D} = \mathcal{D}(\Omega)$. The new symbols are defined implicitly by duality, e.g., $\mathcal{D}'^m = (C_c^0)'$ and $\mathcal{E}'^m = (C^0)'$.

Basic Operations on \mathcal{D}'

X,Y topological vector spaces. $Z \subset X'$ dense subspace. $T: Z \to Y'$ linear, $T': Y \to X$ continuous linear such that

$$\langle Tx, y \rangle = \langle x, T'y \rangle, \quad x, \in \mathbb{Z}, y \in Y$$

Define $\tilde{T}: X' \to Y'$ by $\tilde{T} = (T')'$, i.e., by

$$\langle \tilde{T}x, y \rangle = \langle x, T'y \rangle, \quad x \in X', y \in Y.$$

Obviously, $\tilde{T} = T$ on Z. \tilde{T} is continuous. From density (Y' is Hausdorff), \tilde{T} is the unique continuous extension of T.

Differentiation

 $X = Y = \mathcal{D}(\Omega). \ T = \partial^{\alpha}. \ Z = \mathcal{D}.$

$$\langle \partial_j u, \phi \rangle = \int \partial_j u \cdot \phi = -\int u \partial_j \phi = -\langle u, \partial_j \phi \rangle, \quad u, \phi \in \mathcal{D}(\Omega).$$

then,

$$<\partial^{\alpha}, \phi>=(-1)^{|\alpha|} < u, \partial^{\alpha}\phi> \Longrightarrow \ T'=(-1)^{|\alpha|}\partial^{\alpha}$$

For $u \in \mathcal{D}'(\Omega)$, define $\langle \partial^{\alpha} u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^{\alpha} \phi \rangle$, for all $\phi \in \mathcal{D}(\Omega)$.

Fact: $\mathcal{D} \hookrightarrow \mathcal{D}'(\Omega)$ is dense.

Multiplication by Smooth Functions

 $a \in C^{\infty}(\Omega), \ u, \phi \in \mathcal{D}(\Omega),$

$$\langle au, \phi \rangle = \langle u, a\phi \rangle$$

for $u \in \mathcal{D}'(\Omega)$ define $\langle au, \phi, u, a\phi \rangle$. Using these two operations, $\sum_{\alpha} a_{\alpha} \partial^{\alpha}$ can be defined. Moreover, by this duality approach

- Change of variables for distributions can be defined.
- Fourier transforms and convolutions can be defined.

Support of Distributions

Definition 2. We say $u \in \mathcal{D}'(\Omega)$ vasnishs on $\omega \subset \Omega$ if $u(\phi) = 0$ for all $\phi \in \mathcal{D}(\Omega)$ with supp $\phi \subset \omega$. **Theorem 1.** Let \mathcal{U} be open cover of Ω . $u_1, u_2 \in \mathcal{D}'(\Omega)$ satisfying,

$$u_1\big|_{\omega} = u_2\big|_{\omega}, \quad \forall \omega \in \mathcal{U}$$

Then, $u_1 = u_2$.

Proof. Let $\{\chi_k\}$ be a partition of unity subordinate to \mathcal{U} .

$$u_1(\phi) = u_1\left(\sum_{k \in I} \chi_k \phi\right) = \sum_k u_1(\chi_k \phi) = \sum_k u_2(\chi_k \phi) = u_2(\phi).$$

References

[RUDIN] Walter Rudin, Functional Analysis, McGraw-Hill Inc. Second Edition (1991).