Lecture[1] 6

Last time

$$T_u(\phi) = \int_\Omega u\phi, \qquad j : u \mapsto T_u : C(\Omega) \to RM(\Omega).$$

For K compact supporting ϕ, $\phi \in C^0_c(\Omega)$, the following estimate is used to prove continuity:

$$|T_u(\phi)| \leq \|u\|_{L^1(K)} \cdot \|\phi\|_{C^0(K)} \leq |K| \|u\|_{C^0(K)} \|\phi\|_{C^0(K)}$$

- $T_u : C^0_c(\Omega) \to \mathbb{R}$ continuous, $T_u \in RM(\Omega)$.
- $j : C(\Omega) \to RM(\Omega)$ continuous.
- j injective, $j : C(\Omega) \hookrightarrow RM(\Omega)$ continuous injection.
- $C(\Omega) \hookrightarrow L^1_{loc}(\Omega)$, injection.
- $L^p_{loc} \hookrightarrow L^1_{loc}(\Omega)$, for all $p \in [0, \infty]$.
- j extended to $j : L^1_{loc}(\Omega) \hookrightarrow RM(\Omega)$, continuous injection.

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Injection of Duals

Definition 1. *Let X, Y be topological vector spaces, $A : X \to Y$ linear. We define the* **adjoint** *$A' : X \to Y'$ for $y \in Y'$:*

$$(A'y')(x) = y'(Ax), \qquad \forall x \in X.$$

The following may be written notationally as $< A'y', x > = < y', Ax >$, $A'y' = y' \circ A$, where $A'y'$ is the pullback of y' under $A = A^* y'$.

Equip X', Y' with their weak$*$ topologies, if A continuous, then

$$| < A'y', x > | = | < y', Ax > | \implies A' : X' \to X' \textit{ continuous.}$$

Lemma 1. *$A : X \to Y$ continuous and linear . $A(X)$ dense in $Y \implies A'$ is injective.*

Proof. $\underbrace{(A'y')(x)}_{y'(Ax)} = 0$ for all $x \in X \implies y' : \underbrace{Ax}_{z} \mapsto y'(Ax)$, $z \in A(X)$. □

Corollary 1. *X, Y topological vector space. $j : X \hookrightarrow Y$ continuous injection with a dense image. Then,*

$$j' : Y' \to X'$$

is continuous injection with dense image.

The elements of $\mathcal{D}'^m$ are called *distributions of order* $\leq m$, cf. Figure 1, and the set of all distributions of finite order is denoted by $\mathcal{D}'_F = \bigcup_{m \geq 0} \mathcal{D}'^m$.

[1]Notes by Ibrahim Al Balushi

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\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d] & C_c^m \arrow[r] \arrow[d] & C_c^0 \arrow[d] \arrow[drr] & \text{duality} & RM \arrow[r] & \mathcal{D}'^m \arrow[r] & \mathcal{D}' \\
\mathcal{E} \arrow[r] & C^m \arrow[r] & C^0 \arrow[urr] & \text{“mirror”} & (C^0)' \arrow[r] \arrow[u] & \mathcal{E}'^m \arrow[r] \arrow[u] & \mathcal{E}' \arrow[u]
\end{tikzcd}
$$

Figure 1: Each arrow represents a dense embedding. The domain Ω in each space is understood, e.g., $\mathcal{D} = \mathcal{D}(\Omega)$. The new symbols are defined implicitly by duality, e.g., $\mathcal{D}'^m = (C_c^0)'$ and $\mathcal{E}'^m = (C^0)'$.

Basic Operations on $\mathcal{D}'$

X, Y topological vector spaces. $Z \subset X'$ dense subspace. $T : Z \to Y'$ linear, $T' : Y \to X$ continuous linear such that

$$< Tx, y >=< x, T'y >, \qquad x, \in Z, \ y \in Y.$$

Define $\tilde{T} : X' \to Y'$ by $\tilde{T} = (T')'$, i.e., by

$$< \tilde{T}x, y >=< x, T'y >, \qquad x \in X', \ y \in Y.$$

Obviously, $\tilde{T} = T$ on Z. $\tilde{T}$ is continuous. From density (Y' is Hausdorff), $\tilde{T}$ is the unique continuous extension of T.

Differentiation

$X = Y = \mathcal{D}(\Omega)$. $T = \partial^\alpha$. $Z = \mathcal{D}$.

$$< \partial_j u, \phi >= \int \partial_j u \cdot \phi = - \int u \partial_j \phi = - < u, \partial_j \phi >, \qquad u, \phi \in \mathcal{D}(\Omega),$$

then,

$$< \partial^\alpha, \phi >= (-1)^{|\alpha|} < u, \partial^\alpha \phi > \implies T' = (-1)^{|\alpha|} \partial^\alpha$$

For $u \in \mathcal{D}'(\Omega)$, define $< \partial^\alpha u, \phi >= (-1)^{|\alpha|} < u, \partial^\alpha \phi >$, for all $\phi \in \mathcal{D}(\Omega)$.

Fact: $\mathcal{D} \hookrightarrow \mathcal{D}'(\Omega)$ is dense.

Multiplication by Smooth Functions

$a \in C^\infty(\Omega), \ u, \phi \in \mathcal{D}(\Omega)$,

$$< au, \phi >=< u, a\phi >$$

for $u \in \mathcal{D}'(\Omega)$ define $< au, \phi, u, a\phi >$. Using these two operations, $\sum_\alpha a_\alpha \partial^\alpha$ can be defined. Moreover, by this duality approach

- Change of variables for distributions can be defined.
- Fourier transforms and convolutions can be defined.

Support of Distributions

Definition 2. *We say $u \in \mathcal{D}'(\Omega)$ vasnishs on $\omega \subset \Omega$ if $u(\phi) = 0$ for all $\phi \in \mathcal{D}(\Omega)$ with supp $\phi \subset \omega$.*

Theorem 1. *Let $\mathcal{U}$ be open cover of Ω. $u_1, u_2 \in \mathcal{D}'(\Omega)$ satisfying,*

$$u_1\big|_\omega = u_2\big|_\omega, \qquad \forall \omega \in \mathcal{U}.$$

Then, $u_1 = u_2$.

Proof. Let $\{\chi_k\}$ be a partition of unity subordinate to $\mathcal{U}$.

$$u_1(\phi) = u_1\left(\sum_{k\in I} \chi_k \phi\right) = \sum_k u_1(\chi_k \phi) = \sum_k u_2(\chi_k \phi) = u_2(\phi).$$

□

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References

[RUDIN] Walter Rudin, *Functional Analysis*, McGraw-Hill Inc. Second Edition (1991).