Lecture[1] 5

X Lf-space: $X_1 \subset X_2 \subset \cdots \subset X$ with each X_i Frechet.

Define the local base σ by $A \subset X$ is convex $\to A \in \sigma$ if and only if $\forall n,\ A \cap X_n \in \mathcal{N}(0)$ in X_n.

Fact: X is complete Hausgorff LCTVS such that $\forall n$, the topology of X_n coincides with the one induced by $X_n \subset X$. Proof is in [RUDIN] for the case $X = \mathscr{D}(\Omega)$.
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Test Functions and Distributions: The space $\mathscr{D}(\Omega)$

The space $C_c^\infty(\Omega)$ is said to be the space of *test functions* on Ω, a space consisting of all C^∞ functions whose support is compact in Ω. C_c^∞ also has a vector space structure. A topology may be defined on $C_c^\infty(\Omega)$ by equipping it with the sequence of norms

$$\| \cdot \|_{C^k} = \sup_{x \in \Omega, |\alpha| \leq k} |\partial^\alpha \phi(x)|. \tag{1}$$

Restricting this norm to subspaces $\mathscr{D}(K) = \{\phi \in C_o^\infty(\Omega) : \operatorname{supp} \phi \subset K\}$, $\mathscr{D}_K \subset C_c^\infty(\Omega)$ induces the same Frechet topology on $\mathscr{D}_K$ generated by the family of semi-norm

$$p_k(\phi) = \sup_{x \in K, |\alpha| \leq k} |\partial^\alpha \phi(x)|,$$

By virtue of Theorem[2] 1 in lecture 3, the family of norms in (1) generates a *locally convex metrizable topology* on $C_c^\infty(\Omega)$, however, it ceases to be complete with respect to its metric. In other words the subspaces $\mathscr{D}_K$ do not pass their completeness on to the whole space $C_c^\infty(\Omega)$. Fortunately, the LF-topology we define below will turn out to be complete. However, as we will see it is not metrizable.

We consider the *LF-topology* on $C_c^\infty(\Omega)$ induced by $\mathscr{D}(K_1) \subset \mathscr{D}(K_2) \subset \ldots$, with each $\mathscr{D}(K_n)$ having the Fréchet topology generated by the semi-norms $\{\| \cdot \|_{C^m(K_n)} : m = 1, 2, ...\}$. We formulate this more precisely in the following definition.

Definition 1. *[RUDIN]Def.6.3. Let $\Omega \subset \mathbb{R}^n$ nonempty, open set.*

a) Any compact $K \subset \Omega$, τ_K denotes the Frechet space topology of $\mathscr{D}(K)$ described above; $K_n \subset K_{n+1}$, such that $\bigcup_n K_n = \Omega$.

b) σ be collection of convex sets $A \subset \mathscr{D}(\Omega)$ such that, for all K_n, $A \cap \mathscr{D}_{K_n} \in \mathcal{N}(0)$ in $\mathscr{D}(K_n)$.

σ defines a local base for the LF-topology τ on $C_c^\infty(\Omega)$. Define the topological space

$$\mathscr{D}(\Omega) \overset{def}{=} (C_c^\infty(\Omega), \tau).$$

Definition 2. *X TVS. $\{x_n\}$ is Cauchy if $\forall \mathcal{U} \in \mathcal{N}(0)$ in X $\exists N$ such that $x_n - x_k \in \mathcal{U},\ \forall n, k \geq N$.*

Lemma 1. *Cauchy sequences are bounded.*

[1] Notes by Ibrahim Al Balushi
[2] Theorem 1.37 in Rudin's text

Proof. Let $\mathcal{U}, \mathcal{V} \in \mathcal{N}(0)$ balanced in X satisfying $\mathcal{U} + \mathcal{U} \subset \mathcal{V}$. There exists $N > 0$ such that $x_n - x_N \in \mathcal{U}$. $x_n \in x_n + \mathcal{U}$. Let $s > 1$ such that $x_n \in s\mathcal{U}$,

$$\implies x_n \in s\mathcal{U} + \mathcal{U} \subset s\mathcal{U} + s\mathcal{U} \subset s\mathcal{V}.$$

□

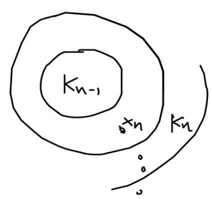
Theorem 1. *[RUDIN] Sec 6.5.*

a) $E \subset \mathscr{D}(\Omega)$ is bounded if and only if $E \subset \mathscr{D}(K)$ for some compact $K \subset \Omega$, and E is bounded in $\mathscr{D}(K)$.

b) $\{\phi_j\} \subset \mathscr{D}(\Omega)$ Cauchy in $\mathscr{D}(\Omega)$ if and only if $\{\phi_j\} \subset \mathscr{D}(K)$, $\exists K \subset \Omega$ compact and ϕ_j is Cauchy in $\mathscr{D}(K)$.

c) $\phi_j \to 0$ in $\mathscr{D}(\Omega)$ if and only if $\{\phi_j\} \subset \mathscr{D}(K)$, $\exists K \subset \Omega$ compact and $\phi_j \to 0$ in $\mathscr{D}(K)$.

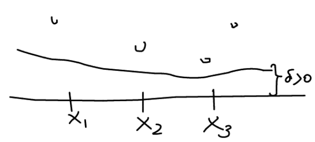
Proof. a) $K_1 \subset K_2 \subset \cdots \subset \Omega$, $\bigcup_n K_n = \Omega$, $E \subset \mathscr{D}(\Omega)$ but $E \not\subset \mathscr{D}(K_n)$ for all n. $\exists x_n \in$



$K_n \backslash K_{n-1}$ $\exists \phi_n \subset E$ such that $\phi_n(x_n) \neq 0$.

$$W = \{\phi \in \mathscr{D}(K) : |\phi(x_n)| < \frac{1}{n}|\phi_n(x_n)|, \ \forall n\}$$

$mW \not\supset E$, for all $m \in \mathbb{N}$. $\phi \in mW \implies |\phi(x_m)| < |\phi_m(x_m)|$. $W \cup \mathscr{D}(K_n)$ such that $\{\phi \in \mathscr{D}(K_n) :$



$\sup_{K_n} |\phi| < \delta\} \subset W \cap \mathscr{D}(K_n)$. E not bounded. E bounded $\implies E \subset \mathscr{D}(K_n)$ $\exists n$.

□

Theorem 2. *Y LCTVS, $f : \mathscr{D}(\Omega) \to Y$ linear. Then the following are equivalent:*

a) f continuous

b) For any $K \subset \Omega$ compact, $f : \mathscr{D}(K) \to Y$ is continuous.

c) $\phi_j \to 0$ in $\mathscr{D}(\Omega) \implies f(\phi_j) \to 0$ in Y.

Proof. *a)* $\iff$ *b)* proved. *a)* $\implies$ *c)* straight forward. For *c)* $\implies$ *a)*: Choose $\mathcal{U} \in \mathcal{N}(0)$ convex and balanced in Y. $V = f^{-1}(\mathcal{U})$ is convex and balanced. $0 \in V$. V open $\iff$ $\mathscr{D}(K) \cap V$ open, for any compact set $K_n \subset \Omega$. □

Definition 3. *An element of $\mathscr{D}'(\Omega)$ is called a* **distribution**. *The dual*

$$RM(\Omega) = [C_c^0(\Omega)]'$$

of $C_c^0(\Omega) = \bigcup_n C_c^0(K_n)$ is called the set of **Radon measures**.

Corollary 1. *The following are equivalent:*

a) f is a distribution.

b) $\forall K \subset \Omega$ compact, $f \in \mathscr{D}'(K) \iff \forall K,\ \exists m$ such that $|f(\phi)| \leq \|\phi\|_{C^m(K)}\ \forall \phi \in \mathscr{D}(K)$.

c) $\phi_j \to 0$ in $\mathscr{D}(K) \implies f(\phi_j) \to 0$.

Definition 4. *Suppose τ_1, τ_2 define two topologies on set X with the property that $\tau_1 \subset \tau_2$, where we do not consider equality. Then τ_1 is said to be a* **weaker** *topology on X.*

Definition 5. *Let X be TVS, X' its dual. The topology on X induced by the semi-norms*

$$\{x \mapsto |f(x)| : f \in X'\},$$

is called the **weak topology** *on X. Similarly, the topology on X' induced by the seminorms*

$$\{f \mapsto |f(x)| : x \in X\},$$

is called the **weak$*$ topology** *on X'.*

Lemma 2. *X' w/ its weak$*$ topology is a Hausdorff LCTVS.*

Proof. Suppose $f \in X',\ f \neq 0$. Explicitly, $\exists x_* \in X$ such that $f(x_*) \neq 0$. $f \mapsto f(x_*)$. □

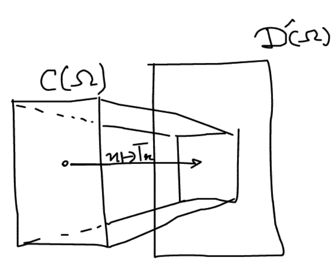
Definition 6. *We equipt $\mathscr{D}'(\Omega)$ with its weak$*$ topology. This topology is generated by the local subbase:*

$$V(\phi, n) = \left\{ f \in \mathscr{D}'(\Omega) : |f(\phi) < \frac{1}{n},\ \phi \in \mathscr{D}(\Omega),\ n \in \mathcal{N} \right\}$$

$f_j \to 0$ in $\mathscr{D}'(\Omega) \iff f_j(\phi) \to 0\ \forall \phi \in \mathscr{D}(\Omega)$.

Notation: $< f, \phi > = < \phi, f > =: f(\phi)$

$u \in C^k$. Define



$$T_u(\phi) = \int_\Omega f\phi \implies T_u \in \mathscr{D}'(\Omega),\ T_u \in RM(\Omega).$$

$C(\Omega) \subset \mathscr{D}'(\Omega)$

$$\int u\phi = \int v\phi \implies u = v$$

$T : u \mapsto T_u : C(\Omega) \to \mathscr{D}'(\Omega)$.

$$\tilde{T}_u(\phi) = \int_\Omega u\phi \; d\mu, \qquad \mu \in RM(\Omega)$$

$T = \tilde{T}$ with $d\mu$ =Lebesque measure.

References

[RUDIN] Walter Rudin, *Functional Analysis*, McGraw-Hill Inc. Second Edition (1991).