Lecture[1] 3

Cont'd Seminorm

Definition 1. *A family of seminorms $\mathcal{P}$ on X is called* **separating** *if*

$$\forall x \in X \backslash \{0\},\ \exists p \in \mathcal{P}\ s.t\ p(x) \neq 0.$$

Lemma 1. *p seminorm.*

a) $p(0) = 0$.

b) $|p(x) - p(y)| \leq p(x - y)$.

c) $p(x) \geq 0$.

d) $\{p(x) = 0\}$ *is a subspace of* X.

e) $B = \{p(x) < 1\}$ *is convex, absorbing, balanced.*

Proof. d) By a), $0 \in B$. For any $\alpha, \beta \in \mathbb{R}$ and $x, y \in B$

$$0 \leq p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) = 0,$$

hence $\alpha x + \beta y \in B$.
e) For convexity we have,

$$p(tx + (1-t)y) \leq tp(x) + (1-t)p(y) < 1.$$

balanced: $|\alpha| < 1$. $p(\alpha x) = |\alpha| p(x) < 1$. Finally, to show absorbance choose $s > p(y)$. $x = \frac{y}{s}$,

$$\implies p(x) = \frac{1}{s} p(y) < 1.$$

hence $x \in B$, thus every $x \in X$ may be scaled down to B. Considering the process in reverse yields the result required. $\square$

Theorem 1. *[**?**][1.37] Let $\mathcal{P}$ be a separating family of seminorms on X. Define*

$$V(p, n) = \{x \in X : p(x) < \frac{1}{n}\} \qquad p \in \mathcal{P}, n \in \mathbb{N}.$$

$$\sigma = \{finite\ intersections\ of\ V(p, n)\}.$$

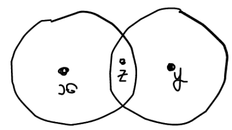
Then σ is a convex balanced local base for a topology τ on X, which turns X into a locally convex Hausdorff TVS such that

a) $p \in \mathcal{P}$ *continuous.*

b) $E \subset X$ *bounded if and only if each* $p \in \mathcal{P}$ *is bounded on* E.

Proof. $A \subset X$ open $\implies$ A is the union of translates of elements from σ.

i) translates of σ cover X.



ii) $A, B \in$ translates of $\sigma \implies z \in A \cap B$ $\exists C$ translate of σ s.t $z \in C \subset A \cap B$.

$$p_1(z-x) < \delta,\ p_2(z-y) < \delta.$$

$$p_1(t-z) < \epsilon \implies p_1(t-x) < \delta$$

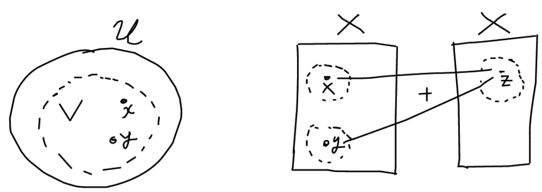
$$p_1(t-x) \le p_1(t-z) + \underbrace{p_1(x-z)}_{=r} \le \epsilon + r$$

$\epsilon < \delta - r.$ $\implies$ τ is locally convex topology. Now, let $x \in X \backslash \{0\}, \exists p \in \mathcal{P}$ $p(x) > 0$ $V(p,n)$ with $\frac{1}{n} < t$ is an open neighbourhood of 0 not including x. We aim to assert a); that is to show continuity. For addition: $+ : X \times X \to X$, choose $\mathcal{U} \in \mathcal{N}(0)$, such that

$$\mathcal{U} \supset V(p_1, n_1) \cap \cdots \cap V(p_k, n_k)$$

Now take

$$V = V(p_1, 2n_1) \cap \cdots \cap V(p_k, 2n_k)$$



Clearly if $x, y \in V$, then $x + y \in \mathcal{U}$ hence $V + V \subset \mathcal{U}$. More generally,

$$(x+V) + (y+V) \subset z + \mathcal{U} \implies + : \mathbb{R} \times X \to X \textit{ continuous.}$$

$p : X \to \mathbb{R}$. $|p(x) - p(y)| < \epsilon$

$$|p(x) - p(y)| \le p(x-y) < \epsilon$$

$x \in y + V(p,n)$: with $\frac{1}{n} < \epsilon$

$$b)(\implies)\textit{bounded}$$

$p \in \mathcal{P}$. $\exists t > 0$ $s.t$ $E \subset tV(p,1)$, $x \in E :$ $p(\frac{x}{t}) < 1 \Leftrightarrow p(x) < t.$ □

Example $X = \mathbb{R}^2,\ p_k(x) = |x_k|$..
$C(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, nonempty domain. K compactly embedded in Ω compact. $p_k(f) = \sup_{x \in K} |f(x)|$

$$K_1 \subset K_2 \subset \cdots \subset \Omega \qquad \textit{compact}$$

[1]Notes by Ibrahim Al Balushi

$$K_i \subset K_{i+1}, \qquad \bigcup_i K_i = \Omega.$$

$$K_i = \{x \in \Omega : dist(x, \partial\Omega) \geq \frac{1}{i}\} \bigcap \overline{B}_i(0)$$

$p_i(f) = \sup_{x \in K_i} |f(x)|$. $\forall K,\ \exists K_i \supset K$. $p_k(f) \leq p_i(f)$.
$V(p_i, k)$

Theorem 2. *Suppose X is a TVS whose topology τ is generated by countable separable family of seminorms $\mathcal{P}$. Then X is a metrizable with metric*

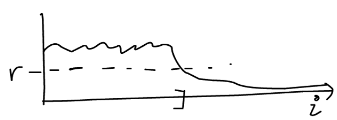
$$d(x, y) = \max_i \frac{\alpha_i p_i(x-y)}{1 + p_i(x-y)},$$

$(\alpha_i > 0,\ \alpha_i \to 0)$. $\mathcal{P} = \{p_1, p_2, ...\}$.

Proof. d is a metric (exercise). **Claim**:

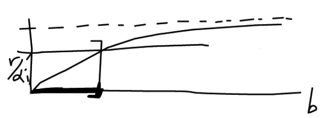
$$B_r = \{x \in X : d(0, x) < r\} \text{ induces a local base for } \tau.$$

Fix $r > 0$.



$$d(0, x) < r \Leftrightarrow \frac{p_i(x)}{1 + p_i(x)} < \frac{r}{\alpha_i}$$

for finitely many i. In other words, collect all i such that $\alpha_i > r$. If $\alpha_i \leq r \implies \frac{\alpha_i b}{1+b} < r,\ \forall b \geq 0$



$$B_r = V(p_1, n_1) \cap \cdots \cap V(p_k, n_k)$$

$\square$

Example: $C(\Omega)$ is metrizable. Moreover, it is complete, locally convex (Frechet). A theorem tells us that: normable = LB + LC. $C(\Omega)$ is not LB hence not normable.
$C^k(\Omega),\ \mathcal{S}(\mathbb{R}^n),\ \mathcal{O}(\Omega)$-holomorphic functions on open sets.

Example: $C^\infty(\Omega) = \mathcal{E}(\Omega)$. $\mathcal{D}(K) = C_o^\infty(K)$.

$$\mathcal{D}_k = \{f \in C^\infty(\Omega) : supp\ f \subseteq K\}$$

$$p_n(f) = \max_{|\alpha| \leq n} \sup_{x \in K_n} |\partial^\alpha f(x)|$$

$\mathcal{D}_k \subset \mathcal{E}(\Omega)$ subspace closed. $\mathcal{E}(\Omega)$ has the Heine-Borel property. We know: LB+HB $\implies$ $dim < \infty \implies$ not LB and not normable.