Lecture[1] 13

Applications of Distributions to Constant Coefficient Operators

Consider the polynomial $P(\xi) = \sum_\alpha a_\alpha \xi^\alpha$ whose coefficients a_α are all constant. We have previously established that
$$\widehat{P(\partial)u}(\xi) = P(i\xi)\widehat{u}(\xi).$$

Definition 1. *$\widehat{P}(\xi) = P(i\xi)$ is called the* ***symbol*** *of $P(\partial)$.*

- If q is a symbol, its associated differential operator is
$$q(D) = q(-i\partial) =: Q(\partial),$$
where $D = -i\partial$, so that
$$\widehat{Q}(\xi) = q(-i \cdot i\xi) = q(\xi).$$

- The *principal symbol* P_m is the homogeneous polynomial
$$\widehat{P}_m(\xi) = \sum_{|\alpha|=m} a_\alpha (i\xi)^\alpha, \tag{1}$$
with m equal to the degree of P. So the principal symbol of $P(D)$ is
$$P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha.$$

Solvability of $P(D)u = f$

- In $\mathscr{D}'$ it is proven (independently) by Malgrange and Ehrenpreis in 1953.
- In $\mathcal{S}'$ is its proven by Hörmander, Lojasiewicz and Bernstein.

The idea comes from the fact
$$\begin{aligned} P(\xi)\widehat{u}(\xi) &= \widehat{f}(\xi) \quad \text{by the Fourier transform} \\ \text{Def: } Z &= \{\xi : P(\xi) = 0\} \\ r(\xi) &= \frac{\widehat{f}(\xi)}{P(\xi)}, \qquad\qquad \xi \in \mathbb{R}^n \backslash Z, \end{aligned}$$
where we can assume $\widehat{f} \in C(\mathbb{R}^n)$, $|\widehat{f}(\xi)| < C(1+|\xi|)^N$. Keeping this definition of r in mind we define the following notion.

Definition 2. *$\widetilde{r} \in \mathcal{S}'$ is called a* ***regularization*** *of r if*
$$\langle \widetilde{r}, \varphi \rangle = \langle r, \varphi \rangle, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}^n \backslash Z). \tag{2}$$

- If $r \in L^1_{loc}$ then its regularization is straightforward.

[1] Notes by Ibrahim Al Balushi

- If Z is discrete and $|P(\xi)| \geq C\text{dist}(\xi, Z)^b$, we can regularize r following the procedure we have used to establish generalized Laurent series.

- In general we can write

$$Z = \bigcup_k Z_k, \qquad \dim Z_k = k,$$

  we have to prove $|P(\xi)| \geq C\text{dist}(\xi, Z_k)^b$.

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Hypoellipticity

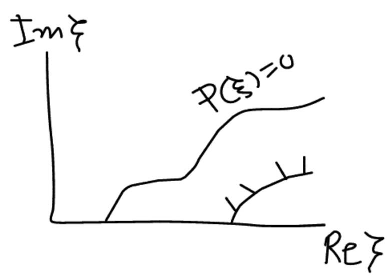
Definition 3. *$P(D)$ is called* ***elliptic*** *if the principal part $P_m(\xi) \neq 0$ for $\xi \neq 0$.*

- Petrowsky proved in 1937 that ellipticity $\iff$ analytic-hypoellipticty.

- Hörmander in 1955 proved the following theorem:

Theorem 1. *$P(D)$ is hypoelliptic if and only if*

$$P(\zeta) = 0,\ |\zeta| \to 0 \implies |Im\zeta| \to \infty.$$



Proof. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain. Define

$$S = \{u \in L^2(\Omega) : P(D)u = 0\}.$$

We claim that S is closed in L^2. Take $u_k \in S$ such that $u_k \to u \in L^2$. Noting that $P(-D)$ is the adjoint of $P(D)$ we have

$$\langle P(D)u, \varphi\rangle = \langle P(D)(u - u_k), \varphi\rangle = \langle u - u_k, P(-D)\varphi\rangle \leq \|u - u_k\|_{L^2}\|P(-D)\varphi\|_{L^2}.$$

Also, $S \subset C^\infty(\Omega)$. Consider $G : S \to \mathbb{R}$, defined by

$$G : u \mapsto \nabla u(0).$$

G is closed, $u_k \to u$, $\nabla u_k(0) \to a$ implies $a = \nabla u(0)$. Thus G is continuous by the closed graph theorem and

$$|\nabla u(0)| \leq C\|u\|_{L^2}, \qquad u \in S. \tag{$*$}$$

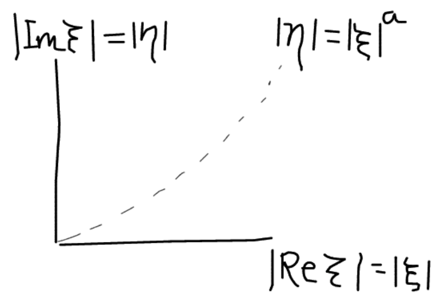
Now, take $u(x) = e^{ix\cdot\zeta}$, $P(\zeta) = 0$. Then

$$P(D)u(x) = P(\zeta)u(x) = 0.$$

and $\nabla u(0) = i\zeta$ while keeping in mind that $\zeta \in \mathbb{C}$. Now using $(*)$

$$\|u\|_{L^2} \leq C_1 e^{C_2|\mathrm{Im}\zeta|} \implies |\zeta| < C_1 e^{C_2|\mathrm{Im}\zeta|}$$

for now if $|\zeta| \to \infty$ then $|\mathrm{Im}\zeta| \to \infty$.



Conversely, we claim

$$\exists a > 0,\ b\ s.t\ |\eta| \leq |\xi|^a \implies |P(\xi + i\eta)| \geq |\xi|^b$$

for ξ large. We prove this claim by considering the set

$$\Sigma = \{t, s : t = |\xi|^2,\ s \geq |\eta|^2 \text{ satisfying } P(\xi + i\eta) = 0\}.$$

$\Sigma \subset \mathbb{R}^{2n+2}$ is semi-algebraic. By Seidenberg-Tarski, the projection Σ' of Σ onto the (t, s)-plane is semi-algebraic. Also, $s \to \infty$ as $t \to \infty$. Define

$$\phi(t) = \min\{s : (t, s) \in \Sigma'\},$$

then for large t, we have $\phi(t) = at^b(1 + o(1))$ and thus $P(\xi + i\eta) = 0$

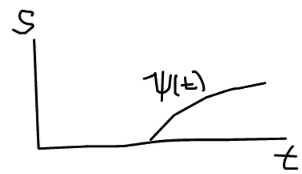
$$\implies |\eta| \geq \sqrt{a}|\xi|^b(1 + o(1))$$

for large ξ. Now define

$$T = \{t = |\xi|^2,\ |\eta|^{2\alpha} \leq t^\beta,\ s \geq |P(\xi + i\eta)|^2\}$$

$\alpha, \beta \in \mathbb{N}$ such that $s \neq 0$ for sufficiently large t. T' is semi-algebraic and thus define

$$\psi(t) = \inf\{s : (t,s) \in T'\} \implies |P(\xi + i\eta)|^2 \geq a + c|\xi|^{2b}(1 + o(1)).$$



We aim to show hypoellipticity. This will be done by constructing a fundamental solution having C^∞ regularity in $\mathbb{R}^n \backslash \{0\}$. Formally,

$$E(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{-\xi \cdot x}}{P(\xi)}\ d\xi.$$

Without loss of generality, assume

$$P(\xi) = \xi_1^m + l.o.t(\text{in } \xi_1), \qquad m = \deg\ P.$$

It is our aim to integrate into the complex plane as to avoid the poles. There exists $A > 0$ such that $P(\xi) \neq 0$ for $|\xi| \geq A$. Define

$$\Gamma_1 = \mathbb{R}^n \backslash [-A, A]^n$$
$$\Gamma_2 = \zeta_2, ..., \zeta_n \in \mathbb{R}^n, \qquad \text{Re}\zeta \in [-A, A]^n$$

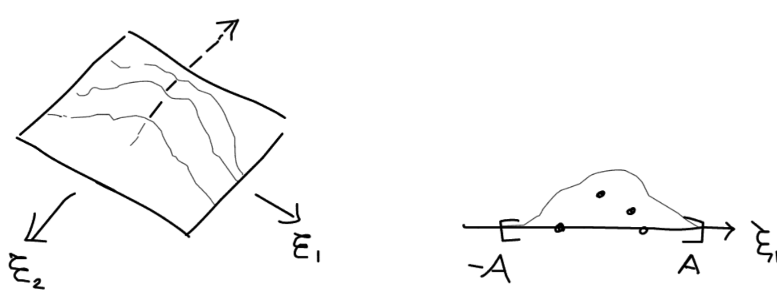


Figure 1: Dots corresponds to zeros of $\zeta_1 \mapsto P(\zeta_1, ..., \zeta_n)$

Define

$$\langle E, \varphi \rangle = (2\pi)^{-n} \int_{\Gamma_1 + \Gamma_2} \frac{\widetilde{\varphi}(\zeta)}{P(\zeta)}\ d\zeta$$

Paley-Wiener tells us $|\widetilde{\widehat{\varphi}}(\zeta)| \leq C(1+|\zeta|)^{-N}$ for $|\mathrm{Im}\zeta|$ bounded. Also,

$$\frac{1}{|P(\zeta)|} \leq C|\mathrm{Re}\zeta|^b.$$

To check if this defines a fundamental solution we carry out the following calculation

$$\langle E, P(-D)\varphi\rangle = (2\pi)^{-n} \int\limits_{\Gamma_1+\Gamma_2} \frac{P(\zeta)\widetilde{\widehat{\varphi}}(\zeta)}{P(\zeta)}\ d\zeta = (2\pi)^{-n} \int\limits_{\Gamma_1+\Gamma_2} \widetilde{\widehat{\varphi}}(\zeta)\ d\zeta = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \widetilde{\widehat{\varphi}}(\zeta)\ d\zeta = \varphi(0)$$

as required. $\square$