Lecture 1

Schwartz Class \( S(\mathbb{R}^n) \)

**Definition 1.** We define the **Schwartz class** functions \( S = S(\mathbb{R}^n) \) by the set
\[
\{ \varphi \in C^\infty(\mathbb{R}^n) : P_{\alpha,\beta}(\varphi) < \infty, \forall \alpha, \beta \}
\]
where
\[
P_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|
\]
defines a family of separating seminorms.

Another way to view the definition above is to consider the space of \( C^\infty \) functions \( \varphi(x) \) satisfying
\[
\left| \frac{\partial^k \varphi(x)}{\partial x^k} \right| \leq C_{m,k} (1 + |x|)^{-m}
\]
for any \( k \) and any positive integer \( m \). As a direct consequence of this definition, Schwartz class functions are \( C^\infty \) functions whose derivatives decay faster than any polynomial. The topology on \( S \) generated by the family of seminorms \( \{P_{\alpha,\beta}\} \) is a Frechet topology. Moreover, the following topological embedding holds
\[
\mathcal{D} \subset S \subset L^1.
\]
In particular, any sequence \( \varphi_n \in \mathcal{D} \) convergent in the topology of \( \mathcal{D} \) is also convergent in the topology of \( S \). Also, \( \mathcal{D} \) is dense in \( S \). This can be easily shown by considering a cut off function \( \chi(x/n) \) to construct a sequence of compactly supported \( C^\infty \) functions converging to a target \( C^\infty \) function which lies in \( S \).

The Fourier Transform

**Definition 2.** Let \( u \in L^1(\mathbb{R}^n) \). The Fourier transform is defined by
\[
\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \int e^{-i\xi \cdot x} u(x) \, dx.
\]
- If \( u \) is continuous then its transform \( \hat{u} \in C_0(\mathbb{R})^n \), due to the Riemann-Lebesgue Lemma.
- If \( \hat{u} \in L^1 \) then,
\[
u(x) = (2\pi)^{-n} \int e^{i\xi \cdot x} \hat{u}(\xi) \, d\xi.
\]

**Theorem 1.** The map \( \mathcal{F} : S \to S \) is an isomorphism with \( \mathcal{F}^{-1} \) given by
\[
\mathcal{F}^{-1} \psi = (2\pi)^{-n} \hat{\varphi}.
\]
moreover, we have
\[
\hat{\partial^\alpha \varphi}(\xi) = (i\xi)^\alpha \hat{\varphi}(\xi) \text{ and } \hat{x^\alpha \varphi} = (i\partial)^\alpha \hat{\varphi}
\]

\(^1\)Notes by Ibrahim Al Balushi
Proof. The proof of this theorem is strictly computational.

\[ \partial^\alpha \hat{\varphi}(\xi) = \int e^{-i\xi \cdot x} (-ix)^\alpha \varphi(x) \, dx \implies \hat{\varphi} \in C^\infty \]

\[ \Rightarrow (-ix)^\alpha = \partial^\alpha \hat{\varphi} \]

\[ \int e^{i\xi \cdot x} \psi(\xi) \hat{\varphi}(\xi) \, d\xi = \int \varphi(y) \, dy \int e^{-i\xi(y-x)} \psi(\xi) \, d\xi \]

\[ = \int \varphi(y) \hat{\psi}(y-x) \, dy = \int \hat{\psi}(y) \varphi(x+y) \, dy. \]

we notice if \( x = 0 \):

\[ \int \hat{\psi} \hat{\varphi} = \int \hat{\psi} \varphi. \]

Now consider the transformation \( \psi(\xi) \mapsto \psi(ex) \) so that \( \hat{\psi}(y) \mapsto e^{-n} \hat{\psi}(y/e) \).

\[ \int e^{-i\xi \cdot x} \psi(ex) \hat{\varphi}(\xi) \, d\xi = \int e^{-n} \hat{\psi}(y/e) \varphi(x+y) \, dy = \int \hat{\psi}(y) \varphi(x+\epsilon) \, dy. \]

Sending \( \epsilon \to 0 \) we obtain

\[ \psi(0) \int e^{-i\xi \cdot x} \hat{\varphi}(\xi) \, d\xi = \varphi(x) \int \hat{\psi}(y) \, dy. \]

Take \( \psi(x) = e^{-|x|^2} \) we obtain the constant \((2\pi)^{-n}\). This rises from the Gaussian integral.

Facts:

- Parseval’s Formula.
- \( \int u \overline{v} = (2\pi)^{-n} \int \hat{u} \hat{v} \).
- \( \hat{u} \overline{\hat{v}} = \hat{u} \cdot \hat{v} \).
- \( \hat{u} \cdot \hat{v} = (2\pi)^{-n} \hat{u} \star \hat{v} \).

Tempered Distributions \( S' \)

**Definition 3.** Linear continuous functional on Schwartz class \( S \) is called a **tempered distribution**. The linear space of tempered distributions is denoted by \( S' \).

We have seen the embedding relation between test functions \( \mathcal{D} \) and Schwartz class function \( S \). Thus any tempered distribution is also a linear continuous distribution on \( \mathcal{D} \). Particularly, since any \( \varphi_n \in \mathcal{D} \) and \( \varphi_n \to \varphi \) in \( \mathcal{D} \) implies that \( \varphi_n \to \varphi \) in \( S \), then if \( f_n(\varphi) \to f(\varphi) \) for all \( \varphi \in S \) implies \( f_n(\varphi_n) \to f(\varphi) \) for all \( \varphi \in \mathcal{D} \),

\[ S' \subset \mathcal{D}'. \]

The definitions for differentiating tempered distributions and test functions coincide. Moreover, multiplication of element of \( S' \) with smooth functions in the performed similarly as in \( \mathcal{D} \), with the one exception that \( a \in C^\infty \) must also satisfy

\[ \left| \frac{\partial^k a(x)}{dx^k} \right| \leq C_k (1 + |x|)^n_k, \quad \forall k. \]
Recall that distributions may be represented locally as some derivative of a bounded function. A similar theorem holds for tempered distributions.

**Theorem 2.** Any \( f \in S' \) can be represented in the following form:

\[
f = \sum_{|\alpha|=0}^{m_1} \partial_\alpha f_\alpha
\]

where \( f_\alpha \) are regular functionals in \( S' \) corresponding to continuous functions \( f_\alpha(x) \) satisfying the estimates

\[
|f_\alpha(x)| \leq C_\alpha (1 + |x|)^{m_2}
\]

where \( m_1 \) and \( m_2 \) are integers.

We define the Fourier transform for tempered distributions:

**Definition 4.** If \( u \in S' \) then define

\[
\langle \widehat{u}, \varphi \rangle = \langle u, \widehat{\varphi} \rangle, \quad \forall \varphi \in S.
\]

**Theorem 3.** The map \( \mathcal{F} : S' \to S' \) is an isomorphism. Moreover,

\[
\widehat{\widehat{u}} = (2\pi)^{-n} \delta, \quad u \in S'
\]

**Proof.** Let \( \varphi \in S \). Then,

\[
\langle \widehat{\widehat{u}}, \varphi \rangle = \langle u, \widehat{\varphi} \rangle = (2\pi)^{-n} \langle u, \widehat{\varphi} \rangle = (2\pi)^{-n} \langle \widehat{u}, \varphi \rangle.
\]

\[
\Box
\]

**Example 1.** Consider the following computation: For any \( \varphi \in S \),

\[
\langle \delta, \varphi \rangle = \langle \delta, \varphi \rangle = \varphi(0) = \int \varphi(x) \, dx = \langle 1, \varphi \rangle
\]

\[
\implies \widehat{\delta} = 1.
\]

**Example 2.** \( \partial^\alpha \delta(\xi) = (i\xi)^\alpha \).

**Simple application: Generalized Liouville’s theorem**

Suppose \( P(\xi) \neq 0 \) if \( \xi \neq 0 \). For example this holds for the Laplacian or the heat operator. Let \( u \in S' \) that satisfies \( P(\partial)u = 0 \). Then

\[
P(i\xi)\widehat{u}(\xi) = 0
\]

\[
supp \widehat{u} \subset \{0\}
\]

\[
\implies \widehat{u} = \sum_\alpha \partial^\alpha \delta, \quad |\alpha| < \infty
\]

\( u \) is a polynomial.