MATH 580 FALL 2018 PRACTICE PROBLEMS

DECEMBER 7, 2018

1. Let u be a bounded harmonic function in an open set $\Omega \subset \mathbb{R}^n$. Show that
$$|\partial^\alpha u(x)| \leq \frac{C_\alpha}{\mathrm{dist}(x, \partial\Omega)^{|\alpha|}},$$
for all $x \in \Omega$ and all $\alpha \in \mathbb{N}_0^n$, where C_α is a constant that is allowed to depend only on α.
2. Suppose that $u \in C^2(\mathbb{R}^n_+) \cap C(\bar{\mathbb{R}}^n_+)$ be a bounded harmonic function in the upper half space $\mathbb{R}^n_+ = \{x :\in \mathbb{R}^n : x_n > 0\}$, satisfying $u \leq 0$ on $\partial\mathbb{R}^n_+$. Show that $u \leq 0$ in $\mathbb{R}^n_+$.
3. Let $\Omega \subsetneq \mathbb{R}^2$ be a domain, and let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a bounded harmonic function in Ω, such that $u \leq 0$ on $\partial\Omega$. Show that $u \leq 0$ in Ω. Is this result true in higher dimensions?
4. Is there a bounded harmonic function in $\mathbb{D}$ that is not uniformly continuous in $\mathbb{D}$?
5. Give an example of an unbounded harmonic function in $B = B_1$, satisfying
$$|\nabla u(x)| \leq \frac{C}{1 - |x|}, \qquad x \in B,$$
for some constant C.
6. Let u be a harmonic function in $\mathbb{D}$, and suppose that $u(r, 0) = u(r\cos\alpha, r\sin\alpha) = 0$ for all $0 \leq r < 1$, where $0 < \alpha \leq \pi$ is a constant. Show that if α is an irrational multiple of π, then $u \equiv 0$. What happens if α is a rational multiple of π?
7. Give an example of a nontrivial entire harmonic function u in $\mathbb{R}^2$, satisfying $u(t, 1) = u(t, -1) = 0$ for all $t \in \mathbb{R}$. Show that such a function u cannot be a polynomial.
8. Let $\Omega \subset \mathbb{R}^n$ be a domain, let $\omega \subset \Omega$ be open, and let $K \subset \Omega$ be compact. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if u is harmonic in Ω, satisfying $|u| \leq 1$ in Ω and $|u| \leq \delta$ in ω, then $|u| \leq \varepsilon$ on K.
9. Let $u \in C(\bar{\mathbb{R}}^2_+)$ be a bounded harmonic function in the upper half plane $\mathbb{R}^2_+$, satisfying $u(x, 0) \to \pi$ as $x \to \infty$ and $u(x, 0) \to 0$ as $x \to -\infty$. Compute the limit of $u(r\cos\theta, t\sin\theta)$ as $r \to \infty$, for each $0 < \theta < \pi$.
10. A positive harmonic function in $\mathbb{R}^2 \setminus \{0\}$ is constant.
11. Let $\Omega \subset \mathbb{R}^3$ be the unit ball with a line L going through the origin removed. In the context of the Dirichlet problem, is the origin regular for Ω?
12. Let $\Omega \subset \mathbb{R}^3$ be the unit ball with the half plane $\{x_2 > 0,\ x_3 = 0\}$ removed. Is every point of $\partial\Omega$ regular?
13. Give an example of a bounded domain with C^1 boundary that does not satisfy the exterior sphere condition at some of its boundary points.
14. Show that a domain with Lipschitz boundary satisfies the exterior cone condition at each of its boundary points.
15. Let $\Omega \subset \mathbb{R}^n$ be an open set.
    (a) Using Green's first identity, show that
    $$\|\nabla u\|^2_{L^2(\Omega)} \leq \varepsilon\|\Delta u\|^2_{L^2(\Omega)} + \tfrac{1}{4\varepsilon}\|u\|^2_{L^2(\Omega)},$$
    for any $\varepsilon > 0$ and $u \in H^2_0(\Omega)$, where $H^2_0(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in $H^2(\Omega)$.
    (b) Under additional assumptions on Ω, and by employing an extension result, show that
    $$\|\nabla u\|^2_{L^2(\Omega)} \leq \varepsilon|u|^2_{H^2(\Omega)} + C\varepsilon^{-1}\|u\|^2_{L^2(\Omega)},$$

for any $\varepsilon > 0$ and $u \in H^2(\Omega)$, with $C > 0$ possibly depending on Ω.

16. Let $I = (a, b)$. Prove the following.
   (a) For any $\varepsilon > 0$, there exists a constant $C = C_\varepsilon$ such that
   $$\|u'\|_{L^2(I)} \leq \varepsilon \|u''\|_{L^2(I)} + C\varepsilon^{-1}\|u\|_{L^2(I)},$$
   for any $u \in C^\infty(I)$.
   (b) Let $u \in L^2(I)$, and suppose that $u'' \in L^2(I)$ exists in the weak sense, i.e., there is $f \in L^2(I)$ such that
   $$\int_I u\varphi'' = \int_I f\varphi \qquad \text{for all} \quad \varphi \in \mathscr{D}(I).$$
   Then $u' \in L^2(I)$ exists in the weak sense.
17. We say that $f \in C^\infty(\Omega)$ is in the *Gevrey class* $G^\alpha(\Omega)$ with $\alpha \geq 1$, if for any ball B with $\bar{B} \subset \Omega$, there exist $\delta > 0$ and $M < \infty$ such that
   $$\|f\|_{C^m(B)} \leq M\frac{(m!)^\alpha}{\delta^m} \qquad \text{for all } m \in \mathbb{N}. \tag{1}$$
   We have $G^\alpha(\Omega) \subset G^\beta(\Omega)$ for $\alpha \leq \beta$, and $G^1(\Omega) = C^\omega(\Omega)$. Also, it makes sense to define $G^\infty = C^\infty$. Hence in some sense, the Gevrey classes fill the gap between C^ω and C^∞. Prove that if $f \in G^\alpha(\Omega)$ for some $\alpha \geq 1$ then any weak solution $u \in H^1_{\text{loc}}(\Omega)$ of $-\Delta u + tu = f$ satisfies $u \in G^\alpha(\Omega)$.
18. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\, 0 < y < x^4\}$.
   (a) Exhibit a function $u \in H^2(\Omega)$ such that $u \notin L^\infty(\Omega)$.
   (b) Is there a function $u \in H^2(\Omega) \cap C_b(\Omega)$, that cannot be extended to $u \in C(\bar{\Omega})$?
19. In each case, exhibit an unbounded function u in Ω, such that $u \in H^k(\Omega)$ for all $k \geq 0$.
   (a) $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0,\, 0 < y < f(x)\}$, where $f \in C([0, \infty))$ is a nonincreasing positive function satisfying $f(x) \to 0$ as $x \to \infty$.
   (b) $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\, 0 < y < e^{-1/x}\}$.
20. Let $B \subset \mathbb{R}^n$ be an open ball.
   (a) Let $0 < k - \frac{n}{p} < 1$, where $k \geq 0$ is an integer, and $1 \leq p < \infty$. Exhibit a function $u \in W^{k,p}(B)$ such that $u \notin C^{0,\alpha}(B)$ for any $\alpha > k - \frac{n}{p}$.
   (b) Let $k - \frac{n}{p} = 0$ and $p > 1$. Exhibit a function $u \in W^{k,p}(B)$ such that $u \notin L^\infty(B)$.
   (c) Let $k - \frac{n}{p} = 1$ and $p > 1$. Exhibit a function $u \in W^{k,p}(B)$ such that $u \notin C^{0,1}(B)$.
21. Let $I = (-1, 1)$ and $u(x) = |x|$. Show that $u \in W^{1,\infty}(I)$ but u is not in the closure of $C^1(I) \cap W^{1,\infty}(I)$ in $W^{1,\infty}(I)$.
22. Let $u \in C^\infty(\Omega)$ be given in polar coordinates by $u(r, \theta) = r^a \sin(a\theta)$ with
   $$\Omega = \{(r, \theta) : r < 1,\, 0 < \theta < \pi/a\},$$
   where $a \geq \frac{1}{2}$ is a constant. Determine the values of $p \geq 1$ such that $u \in W^{2,p}(\Omega)$.
23. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and consider the bilinear form
   $$a(u, v) = \int_\Omega (a_{ij}\partial_i u \partial_j v + cuv),$$
   where the repeated indices are summer over, and the coefficients a_{ij} and c are smooth functions on $\overline{\Omega}$, with a_{ij} satisfying the uniform ellipticity condition
   $$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \xi \in \mathbb{R}^n, \quad x \in \overline{\Omega},$$
   for some constant $\lambda > 0$.
   (a) Show that the mapping $A : H^1_0(\Omega) \to [H^1_0(\Omega)]'$, defined by $\langle Au, v\rangle = a(u, v)$, is bounded, where $\langle\cdot,\cdot\rangle$ is the duality pairing between $[H^1_0(\Omega)]'$ and $H^1_0(\Omega)$.

(b) Show that if $c \geq 0$ then

$$\langle Au, u\rangle \geq \alpha \|u\|_{H^1}^2, \qquad u \in H_0^1(\Omega),$$

for some constant $\alpha > 0$. Show also that the inequality is still true (with possibly different $\alpha > 0$) if c is slightly negative.

(c) Supposing that $c \geq 0$, show that given $f \in L^2(\Omega)$, there exists a unique function $u \in H_0^1(\Omega)$ satisfying $a(u, v) = \int_\Omega fv$ for all $v \in H_0^1(\Omega)$.

(d) Suppose that $u \in H_0^1(\Omega)$ is sufficiently smooth and satisfies $a(u, v) = \int_\Omega fv$ for all $v \in H_0^1(\Omega)$. What differential equation does u satisfy in Ω? Is $u = 0$ on $\partial\Omega$?

24. In the setting of the preceding problem, let $u \in H_0^1(\Omega)$ satisfy

$$a(u, v) = \int_\Omega fv \qquad \text{for all } v \in H_0^1(\Omega)$$

where $f \in H^k(\Omega)$ with some $k \geq 0$.

(a) Show that $u \in H_{\text{loc}}^{k+2}(\Omega)$.

(b) Prove that $u \in H^{k+2}(\Omega)$.

25. Let $\Omega \subset \mathbb{R}^n$ be an open set such that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Show that the first Neumann eigenvalue of Ω is $\lambda_1 = 0$, and the dimension of the eigenspace corresponding to this eigenvalue (i.e., the multiplicity of λ_1) is equal to the number of connected components of Ω. Can Ω have infinitely many connected components?

26. Prove that there exists a function $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \qquad \text{for all } v \in H^1(\Omega),$$

if and only if $\int f = 0$. Show that such a function is unique up to an additive constant.

27. Design a weak formulation of the Robin problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_\nu u + u = g & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain with C^1 boundary, and f and g are given functions. Prove that a unique weak solution exists, under suitable conditions on the data.

28. Let $L_{\text{per}}^2(\mathbb{R}) = \{f \in L_{\text{loc}}^2(\mathbb{R}) : \tau_{2\pi}^* f = f\}$, where τ_h is the translation operator $\tau_h(x) = x + h$, and let $H_{\text{per}}^k(\mathbb{R}) = H_{\text{loc}}^k(\mathbb{R}) \cap L_{\text{per}}^2(\mathbb{R})$. Prove the following.

(a) $H_{\text{per}}^k(\mathbb{R})$ is a Hilbert space for each $k \geq 0$, with $H_{\text{per}}^0(\mathbb{R}) = L_{\text{per}}^2(\mathbb{R})$, and that

$$\langle u, v\rangle_{L^2} = \int_a^b uv, \qquad \text{and} \qquad \langle u, v\rangle_{H^k} = \int_a^b \big(uv + u^{(k)} v^{(k)}\big), \tag{2}$$

are inner products in $L_{\text{per}}^2(\mathbb{R})$ and in $H_{\text{per}}^k(\mathbb{R})$, respectively, whenever $b - a \geq 2\pi$.

(b) $C_{\text{per}}^\infty(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f(x) = f(x + 2\pi)\}$ is dense in $H_{\text{per}}^k(\mathbb{R})$ for each $k \geq 0$.

(c) The embedding $H_{\text{per}}^1(\mathbb{R}) \hookrightarrow L_{\text{per}}^2(\mathbb{R})$ is compact.

29. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $V = H_0^1(\Omega)$ or $V = H^1(\Omega)$, depending on the type of boundary condition we wish to impose. Assume that the Laplacian $\Delta : V \to V'$ has a compact resolvent in $L^2(\Omega)$, and denote by $\lambda_1 \leq \lambda_2 \leq \ldots$ the eigenvalues of $-\Delta$. Show that

$$\lambda_k = \max_{X \in \Phi_{k-1}} \inf_{u \in X^\perp} \frac{\|\nabla u\|_{L^2(\Omega)^2}}{\|u\|_{L^2(\Omega)^2}}, \tag{3}$$

where $\Phi_m = \{X \subset V \text{ linear subspace} : \dim X = m\}$ is the m-th Grassmannian of V, and $X^\perp$ is understood as $\{u \in V : u \perp_{L^2} X\}$.

30. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^2 boundary. By using Hopf's boundary point lemma, prove that the solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \partial\Omega, \end{cases}$$

is unique up to an additive constant.

31. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, such that $P(\Omega) = \Omega$, where $P : (x', x_n) \mapsto (x', -x_n)$ is the reflection through the hyperplane $\{x_n = 0\}$.
    (a) Show that the first Dirichlet eigenfunction of Ω is symmetric with respect to $\{x_n = 0\}$.
    (b) If Ω is a ball, show that the first Dirichlet eigenfunction is spherically symmetric.