PERRON'S METHOD

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Abstract. We present here the classical method of subharmonic functions for solving the Dirichlet problem that culminated in the works of Perron and Wiener.

Contents

1. Brief history of the Dirichlet problem

Given a domain $\Omega \subset \mathbb{R}^n$ and a function $g : \partial\Omega \to \mathbb{R}$, the *Dirichlet problem* (for the Laplace equation) is to find a function u satisfying

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In the preceding chapter, we have established that uniqueness holds if Ω is bounded and g is continuous. We have also seen that the Dirichlet problem has a solution if Ω is a ball.

The Dirichlet problem turned out to be fundamental in many areas of mathematics and physics, and the efforts to solve this problem led directly to many revolutionary ideas in mathematics. The importance of this problem cannot be overstated.

The first serious study of the Dirichlet problem on general domains with general boundary conditions was done by George Green in his *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, published in 1828. As we have seen in the previous set of notes, he reduced the problem into a problem of constructing what we now call Green's functions, and argued that Green's function exists for any domain. His methods were not rigorous by today's standards, but the ideas were highly influential in the subsequent developments. Most probably, Green never knew the real importance of his discovery, as the Essay went unnoticed by the community until 1845, four years after Green's death, when William Thomson rediscovered it.

The next idea came from Gauss in 1840. He noticed that given a function ρ on $\partial\Omega$, the single layer potential

$$u(y) = (V\rho)(y) \equiv \int_{\partial\Omega} E_y \rho, \tag{2}$$

is harmonic in Ω, and hence that if we find ρ satisfying $V\rho = g$ on $\partial\Omega$, the Dirichlet problem would be solved. Informally, we want to arrange electric charges on the surface $\partial\Omega$ so that

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the resulting electric potential is equal to g on $\partial\Omega$. If we imagine that $\partial\Omega$ is made of a good conductor, then in the absence of an external field, the equilibrium configuration of charges on the surface will be the one that produces constant potential throughout $\partial\Omega$. The same configuration also minimizes the electrostatic energy

$$E(\rho) = \frac{1}{2}\int_{\partial\Omega} \rho V\rho, \tag{3}$$

among all ρ such that the net charge $\int_{\partial\Omega}\rho$ is fixed. In order to solve $V\rho = g$, we imagine that there is some external electric field whose potential at the surface coincides with $-g$. The equilibrium configuration in this case would satisfy $V\rho - g = \text{const}$, and minimize the energy

$$E(\rho) = \frac{1}{2}\int_{\partial\Omega} \rho V\rho - \int_{\partial\Omega} g\rho, \tag{4}$$

among all ρ such that the net charge is fixed. Then we would have $V(\rho - \rho') = g$ for some ρ' satisfying $V\rho' = \text{const}$, or more directly, we can simply add a suitable constant to $u = V\rho$ to solve the Dirichlet problem. Gauss did not prove the existence of a minimizer to (4), but he remarked that it was obvious.

Around 1847, that is just after Green's work became widely known, William Thomson (Lord Kelvin) and Gustav Lejeune-Dirichlet suggested to minimize the energy

$$E(u) = \int_{\Omega} |\nabla u|^2, \tag{5}$$

subject to $u|_{\partial\Omega} = g$. Note that by Green's first identity we have

$$E(u) = \int_{\partial\Omega} u\partial_\nu u - \int_{\Omega} u\Delta u, \tag{6}$$

which explains why $E(u)$ can be considered as the energy of the configuration, since in view of Green's representation formula

$$u(y) = \int_{\Omega} E_y \Delta u + \int_{\partial\Omega} u\partial_\nu E_y - \int_{\partial\Omega} E_y \partial_\nu u, \tag{7}$$

$\partial_\nu u$ is the surface charge density, and $-\Delta u$ is the volume charge density, that produce the field u. This and other considerations seemed to show that the Dirichlet problem is equivalent to minimizing the energy $E(u)$ subject to $u|_{\partial\Omega} = g$. Moreover, since $E(u) \geq 0$ for any u, the existence of u minimizing $E(u)$ appeared to be obvious. Riemann called these two statements the *Dirichlet principle*, and used it to prove his fundamental mapping theorem, in 1851. However, starting around 1860, the Dirichlet principle in particular and calculus of variations at the time in general went under serious scrutiny, most notably by Karl Weierstrass and Riemann's former student Friedrich Prym. Weierstrass argued that even if E is bounded from below, it is possible that the infimum is never attained by an admissible function, in which case there would be no admissible function that minimizes the energy. He backed his reasoning by an explicit example of an energy that has no minimizer. Let us look at his example.

Example 1 (Weierstrass 1870)**.** Consider the problem of minimizing the energy

$$Q(u) = \int_I x^2 |u'(x)|^2 \mathrm{d}x, \tag{8}$$

for all $u \in \mathscr{C}(\bar{I})$ with piecewise continuous derivatives in I, satisfying the boundary conditions $u(-1) = 0$ and $u(1) = 1$, where $I = (-1, 1)$. The infimum of E over the admissible functions is

0, because obviously $E \geq 0$ and for the function

$$v(x) = \begin{cases} 0 & \text{for } x < 0, \\ x/\delta & \text{for } 0 < x < \delta, \\ 1 & \text{for } x > \delta, \end{cases} \tag{9}$$

we have $E(v) = \frac{\delta}{3}$, which can be made arbitrarily small by choosing $\delta > 0$ small. However, there is no admissible function u for which $E(u) = 0$, since this would mean that $u(x) = 0$ for $x < 0$ and $u(x) = 1$ for $x > 0$.

In 1871, Prym constructed a striking example of a continuous function g on the boundary of a disk, such that there is not a single function u with finite energy that equals g on the boundary. This makes it impossible even to talk about a minimizer since all functions with the correct boundary condition would have infinite energy.

Now that the Dirichlet principle is not reliable anymore, it became an urgent matter to solve the Dirichlet problem to "rescue" the Riemann mapping theorem. By 1870, Weierstrass' former student Hermann Schwarz had largely succeeded in achieving this goal. He solved the Dirichlet problem on polygonal domains by an explicit formula, and used an iterative approximation process to extend his results to an arbitrary planar region with piecewise analytic boundary. His approximation method is now known as the *Schwarz alternating method*, and is one of the popular methods to solve boundary value problems on a computer.

The next advance was Carl Neumann's work of 1877, that was based on an earlier work of August Beer from 1860. The idea was similar to Gauss', but instead of the single layer potential, Beer suggested the use of the double layer potential

$$u(y) = (K\mu)(y) \equiv \int_{\partial\Omega} \mu \partial_\nu E_y. \tag{10}$$

The function u is automatically harmonic in Ω, and the requirement $u|_{\partial\Omega} = g$ is equivalent to the integral equation $(1 - 2K)\mu = 2g$ on the boundary. This equation was solved by Neumann in terms of the series

$$(1 - 2K)^{-1} = 1 + 2K + (2K)^2 + \ldots, \tag{11}$$

which bears his name now. Neumann showed that the series converges if Ω is a 3 dimensional convex domain whose boundary does not consist of two conical surfaces. The efforts to solve the equation $(1 - 2K)\mu = 2g$ in cases the above series does not converge, led Ivar Fredholm to the discovery of Fredholm theory in 1900.

Since the analyticity or convexity conditions on the boundary seemed to be rather artificial, the search was still on to find a good method to solve the general Dirichlet problem. Then in 1887, Henri Poincaré published a paper introducing a very flexible method with far reaching consequences. Poincaré started with a subharmonic function that has the correct boundary values, and repeatedly solved the Dirichlet problem on small balls to make the function closer and closer to being harmonic. He showed that the process converges if the succession of balls is chosen carefully, and produces a harmonic function in the interior. Moreover, this harmonic function assumes correct boundary values, if each point on the boundary of the domain can be touched from outside by a nontrivial sphere. The process is now called Poincaré's *sweeping out process* or the *balayage method*.

Poincaré's work made the Dirichlet problem very approachable, and invited further work on weakening the conditions on the boundary. For instance, it led to the work of William Fogg Osgood, published in 1900, in which the author establishes solvability of the Dirichlet problem in very general planar domains. While the situation was quite satisfactory, there had essentially been no development as to the validity of the original Dirichlet principle, until 1899, when David Hilbert gave a rigorous justification of the Dirichlet principle under some assumptions on the boundary of the domain. This marked the beginning of a major program to put calculus of variations on a firm foundation.

During that period it was generally believed that the assumptions on the boundary of the domain that seemed to be present in all available results were due to limitations of the methods employed, rather than being inherent in the problem. However, the following example due to Stanisław Zaremba showed that there exist regions in which the Dirichlet problem is not solvable, even when the boundary condition is completely reasonable.

Example 2 (Zaremba 1911)**.** Let $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disk, and consider the domain $\Omega = \mathbb{D} \setminus \{0\}$. The boundary of Ω consists of the circle $\partial\mathbb{D}$ and the point $\{0\}$. Consider the Dirichlet problem $\Delta u = 0$ in Ω, with the boundary conditions $u \equiv 0$ on $\partial\mathbb{D}$ and $u(0) = 1$. Suppose that there exists a solution. Then u is harmonic in Ω, and continuous in $\mathbb{D}$ with $u(0) = 1$. Since u is bounded in Ω, invoking the removable singularity theorem, we can extend u continuously to $\mathbb{D}$ so that the resulting function is harmonic in $\mathbb{D}$. By uniqueness of solutions to the Dirichlet problem in $\mathbb{D}$, the extension must identically be equal to 0, because $u \equiv 0$ on $\partial\mathbb{D}$. However, this contradicts the fact that u is continuous in $\mathbb{D}$ with $u(0) = 1$. Hence there is no solution to the original problem. The boundary condition at $x = 0$ is simply "ignored".

One could argue that Zaremba's example is not terribly surprising because the boundary point 0 is an isolated point. However, in 1913, Henri Lebesgue produced an example of a 3 dimensional domain whose boundary consists of a single connected piece. This example will be studied in §4, Example 15. The time period under discussion is now 1920's, which saw intense developments in the study of the Dirichlet problem, then known as potential theory, powered by the newly founded Lebesgue integration theory and functional analytic point of view. Three basic approaches came out of this: Poincaré-type methods which use subharmonic functions, integral equation methods based on potential representations of harmonic functions, and finally, variational methods related to minimizing the Dirichlet energy. In this chapter, we will study the method of subharmonic functions in detail.

2. Families of harmonic functions

As a rule, the solution to a general Dirichlet problem is constructed by some type of infinite process. In particular, results on the convergence of sequences of harmonic functions will be of importance. We start here with two fundamental convergence theorems for sequences of harmonic functions, proved by Axel Harnack in 1887. The first of them concerns uniform convergence and can be thought of as an analogue of the Weierstrass convergence theorem from complex analysis. It says that the space harmonic functions is closed under locally uniform convergence.

Theorem 3 (Harnack's first theorem)**.** *Let Ω be an open set, and let $\{u_j\}$ be a sequence of harmonic functions in Ω, that converges locally uniformly in Ω. Then the limit function u is harmonic in Ω. Furthermore, $\partial^\alpha u_j \to \partial^\alpha u$ locally uniformly in Ω for each multi-index α, i.e., $u_j \to u$ in $\mathscr{C}^\infty(\Omega)$.*

Proof. First of all, since $u_j \to u$ locally uniformly, we have $u \in \mathscr{C}(\Omega)$. Moreover, for any $\overline{B_r(y)} \subset \Omega$ and any j, we have

$$u_j(y) = \frac{1}{|B_r|} \int_{B_r(y)} u_j. \tag{12}$$

By the locally uniform convergence, $u_j(y) \to u(y)$ and $\int_{B_r(y)} u_j \to \int_{B_r(y)} u$, which implies that u satisfies the mean value property for every ball whose closure is in Ω. Hence u is harmonic by Koebe's converse of the mean value property. Let $K \subset \Omega$ be a compact set. Then there exists a compact set $K' \subset \Omega$ such that $K \subset K'$ and $r = \text{dist}(K, \partial K') > 0$. Now the derivative estimates for harmonic functions give

$$\sup_K |\partial^\alpha (u_j - u)| \leq |\alpha|! \big(\frac{ne}{r}\big)^{|\alpha|} \sup_{K'} |u_j - u|, \tag{13}$$

which completes the proof. □

Before stating the second theorem of Harnack which deals with nondecreasing sequences of harmonic functions, we prove a generalized version of the Harnack inequality. We emphasize here that the meaning of the following Harnack inequality lies in the fact that the constant C does not depend on the function u.

Theorem 4 (Harnack inequality)**.** *Let Ω be a domain (i.e., connected open set), and let $K \Subset \Omega$ be its compact subset. Then there exists a constant $C > 0$, possibly depending on K, such that for any harmonic and nonnegative function u in Ω, we have*

$$u(x) \leq Cu(y) \qquad x, y \in K. \tag{14}$$

Proof. It suffices to prove the inequality for strictly positive harmonic functions, since if $u \geq 0$ then for any $\varepsilon > 0$ we would have

$$u(x) + \varepsilon \leq Cu(y) + C\varepsilon \qquad x, y \in K, \tag{15}$$

and sending $\varepsilon \to 0$ would establish the claim.

Recall the primitive Harnack inequality: If $\overline{B_{2r}(y)} \subset \Omega$ and $x \in B_r(y)$ then $u(x) \leq 2^n u(y)$. The idea is to piece together primitive Harnack inequalities to connect any pair of points in Ω. One way of doing this would be to simply integrate the differential Harnack inequality. Here we will use a slightly different approach. For $x, y \in \Omega$, define

$$s(x, y) = \sup \big\{ \frac{u(x)}{u(y)} : u > 0, \, \Delta u = 0 \text{ in } \Omega \big\}. \tag{16}$$

First, let us prove that $s(x, y)$ is finite for any $x, y \in \Omega$. Fix $y \in \Omega$, and let $\Sigma = \{x \in \Omega : s(x, y) < \infty\}$. Obviously $y \in \Sigma$, so Σ is nonempty. If $x \in \Sigma$, then $u \leq 2^n u(x)$ in a small ball centred at x, so Σ is open. Moreover, Σ is relatively closed in Ω, because if $\Sigma \ni x_j \to x \in \Omega$ then for sufficiently large j we would have $u(x) \leq 2^n u(x_j)$. We conclude that $\Sigma = \Omega$.

Let K be a compact subset of Ω, and let $r = \frac{1}{4}\text{dist}(K, \partial\Omega)$. Then we can cover K by finitely many sets of the form $B_r(a_k)$, with $a_k \in K$ and say $k = 1, \ldots, N$. This means that for any pair $x, y \in K$, there is a pair of indices j, k taken from $1, \ldots, N$, such that $x \in B_r(a_j)$ and $y \in B_r(a_k)$. We immediately have $u(x) \leq 2^n u(a_j)$ and $u(a_k) \leq 2^n u(y)$, which implies that $u(x) \leq 2^{2n} \big(\max_{j,k} s(a_j, a_k) \big) u(y)$. □

Harnack's second theorem, also known as the *Harnack principle*, can be thought of as the "monotone convergence theorem" for harmonic finctions.

Theorem 5 (Harnack principle)**.** *Let Ω be a domain, and let $u_1 \leq u_2 \leq \ldots$ be a nondecreasing sequence of harmonic functions in Ω. Then either*

- *$u_j(x) \to \infty$ for each $x \in \Omega$, or*
- *$\{u_j\}$ converges locally uniformly in Ω.*

Proof. Suppose that $u_j(y) \leq M$ for some $y \in \Omega$ and $M < \infty$. Obviously, $u_j(y)$ is convergent. Let K be a compact subset of Ω, and without loss of generality, assume that $y \in K$. Then since $u_{j+k} - u_j \geq 0$ for $k > 0$, by the Harnack inequality, there exists $C > 0$ such that for any $x \in K$ we have

$$u_{j+k}(x) - u_j(x) \leq C(u_{j+k}(y) - u_j(y)), \tag{17}$$

which implies that $\{u_j\}$ converges uniformly in K. Since the compact set $K \subset \Omega$ can be chosen arbitrarily, we conclude that the sequence $\{u_j\}$ converges locally uniformly in Ω. □

Before closing this section, we study sequential compactness of bounded families of harmonic functions, in the topology of locally uniform convergence. Such a compactness is customarily called *normaility*. First we recall the Arzelà-Ascoli theorem, in a form that is convenient for our purposes.

Theorem 6 (Arzelà-Ascoli)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $f_j : \Omega \to \mathbb{R}$ be a sequence that is locally equicontinuous and locally equibounded. Then there is a subsequence of $\{f_j\}$ that converges locally uniformly.*

That the sequence $\{f_j\}$ is *locally equibounded* means that for any compact set $K \subset \Omega$ one has $\sup_j \sup_K |f_j| < \infty$. Similarly, that the sequence $\{f_j\}$ is *locally equicontinuous* means that for any compact set $K \subset \Omega$ the sequence $\{f_j\}$ is (uniformly) equicontinuous on K. If $\{f_j\}$ is a sequence of harmonic functions, then the equicontinuity condition can be dropped from the Arzelà-Ascoli theorem, because we can bound derivatives of a harmonic function by how large the function itself is. This is an analogue of Montel's theorem in complex analysis.

Theorem 7. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\{f_j\}$ be a locally equibounded sequence of harmonic functions in Ω. Then there is a subsequence of $\{f_j\}$ that converges locally uniformly.*

Proof. In view of the Arzelà-Ascoli theorem, it suffices to show local equicontinuity of $\{f_j\}$. We will prove here that $\{f_j\}$ is equicontinuous on any closed ball $\bar{B} \subset \Omega$, and the general case follows by a covering argument. Let $B = B_\rho(y)$ and $B' = B_{\rho+r}(y)$ be two concentric balls such that $\bar{B}' \subset \Omega$ and $r > 0$. Then the gradient estimate gives

$$|\nabla f_j(x)| \leq \frac{n}{r} \max_{\partial B_r(x)} |f_j|, \quad \text{for} \quad x \in B, \qquad \text{hence} \qquad \sup_B |\nabla f_j| \leq \frac{n}{r} \sup_{B'} |f_j|.$$

For $x, z \in B$, we have

$$|f_j(z) - f_j(x)| \leq |z - x| \cdot \frac{n}{r} \sup_{B'} |f_j|.$$

Since $\sup_{B'} |f_j|$ is bounded uniformly in j, the sequence $\{f_j\}$ is equicontinuous on B. □

3. Perron's method

In this section, we will discuss the method discovered by Oskar Perron in 1923, as a simpler replacement of the Poincaré process. In a certain sense, everything we have studied about harmonic functions so far will be culminated into the fundamental results of this and the next sections. Recall that we want to solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{18}$$

In what follows, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, and that $g : \partial\Omega \to \mathbb{R}$ is a bounded function. Recall that a continuous function $u \in \mathscr{C}(\Omega)$ is called *subharmonic* in Ω, if for any $y \in \Omega$, there exists $r^* = r^*(y) > 0$ such that

$$u(y) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u, \qquad 0 < r < r^*. \tag{19}$$

Let us denote by $\mathfrak{Sub}(\Omega)$ the set of subharmonic functions on Ω. The following properties will be useful.

- If $u \in \mathfrak{Sub}(\Omega)$ and if $u(z) = \sup_\Omega u$ for some $z \in \Omega$, then u is constant.
- If $u \in \mathfrak{Sub}(\Omega) \cap \mathscr{C}(\bar{\Omega})$, $v \in \mathscr{C}(\bar{\Omega})$ is harmonic in Ω, and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.
- If $u_1, u_2 \in \mathfrak{Sub}(\Omega)$ then $\max\{u_1, u_2\} \in \mathfrak{Sub}(\Omega)$.
- If $u \in \mathfrak{Sub}(\Omega)$ and if $\bar{u} \in \mathscr{C}(\Omega)$ satisfies $\Delta \bar{u} = 0$ in U and $\bar{u} = u$ in $\Omega \setminus U$ for some domain U with $\bar{U} \subset \Omega$, then $\bar{u} \in \mathfrak{Sub}(\Omega)$.

The first two properties are simply the strong and weak maximum principles. The third property is clear from

$$u_i(y) \le \frac{1}{|\partial B_r|}\int_{\partial B_r(y)} u_i \le \frac{1}{|\partial B_r|}\int_{\partial B_r(y)} \max\{u_1,u_2\}, \qquad i=1,2. \tag{20}$$

For the last property, we only need to check (19) for $y \in \partial U$, as

$$\bar u(y) = u(y) \le \frac{1}{|\partial B_r|}\int_{\partial B_r(y)} u \le \frac{1}{|\partial B_r|}\int_{\partial B_r(y)} \bar u, \tag{21}$$

where we have used the weak maximum (or comparison) principle in the last inequality. To proceed further, we define the *Perron (lower) family*

$$S_g = \{v \in \mathfrak{Sub}(\Omega)\cap \mathscr{C}(\bar\Omega) : v|_{\partial\Omega} \le g\}, \tag{22}$$

and the *Perron (lower) solution* $u:\Omega\to\mathbb{R}$ by

$$u(x) = (P_\Omega g)(x) = \sup_{v\in S_g} v(x), \qquad x\in\Omega. \tag{23}$$

Any constant function c satisfying $c \le g$ is in S_g, so $S_g \neq \varnothing$. Moreover, any $v \in S_g$ satisfies $v \le \sup_{\partial\Omega} g$, hence the Perron solution u is well-defined. We will show that the Perron solution is a solution of the Dirichlet problem, under some mild regularity assumptions on the boundary of Ω. Before doing so, let us perform a consistency check. Suppose that $w \in \mathscr{C}^2(\Omega)\cap\mathscr{C}(\bar\Omega)$ satisfies $\Delta w = 0$ in Ω and $w = g$ on $\partial\Omega$. Then obviously $w \in S_g$. Also, the weak maximum principle shows that any $v \in S_g$ satisfies $v \le w$ pointwise. Therefore we must have $u = w$.

Theorem 8 (Perron 1923)**.** *For the Perron solution $u = P_\Omega g$, we have $\Delta u = 0$ in Ω.*

Proof. Let $B = B_r(x)$ be a nonempty open ball whose closure is in Ω, and let $\{u_k\} \subset S_g$ be a sequence satisfying $u_k(x) \to u(x)$ as $k\to\infty$. Without loss of generality, we can assume that the sequence is nondecreasing, e.g., by replacing u_k by $\max\{u_1,\ldots,u_k\}$. For each k, let $\bar u_k \in \mathscr{C}(\Omega)$ be the function harmonic in B which agrees with u_k in $\Omega\setminus B$. We have $u_k \le \bar u_k$, and $\bar u_k \in S_g$ hence $\bar u_k(x) \le u(x)$, so $\bar u_k(x)\to u(x)$ as well. The sequence $\{\bar u_k\}$ is also nondecreasing, so by the Harnack principle, there exists a harmonic function $\bar u$ in B such that $\bar u_k \to \bar u$ locally uniformly in B. In particular, we have $\bar u(x) = u(x)$.

We want to show that $u = \bar u$ in B, which would then imply that u is harmonic in Ω. Pick $y \in B$, and let $\{v_k\}\subset S_g$ be a sequence satisfying $v_k(y)\to u(y)$. Without loss of generality, we can assume that the sequence is nondecreasing, that $\bar u_k \le v_k$, and that v_k is harmonic in B. Again by the Harnack principle, there exists a harmonic function v in $B_r(x)$ such that $v_k \to v$ locally uniformly in B, and we have $v(y) = u(y)$. Because of the arrangement $\bar u_k \le v_k$, we get $\bar u \le v$ in B, and in addition taking into account that $v_k \le u$ and that $\bar u_k(x)\to u(x)$, we infer $v(x) = u(x)$. So $\bar u - v$ is harmonic and nonpositive in B, while $\bar u(x) - v(x) = 0$. Then the strong maximum principle gives $\bar u = v$ in B, which implies that $\bar u(y) = u(y)$. As $y \in B$ was arbitrary, $u = \bar u$ in B. □

Now we need to check if u satisfies the required boundary condition $u|_{\partial\Omega} = g$. Let $z \in \partial\Omega$, and let us try to imagine what can go wrong so that $u(x) \not\to g(z)$ as $x \to z$. It is possible that $\liminf_{x\to z} u(x) < g(z)$, or $\limsup_{x\to z} u(x) > g(z)$, or both. To rule out the first scenario, it suffices to show that there is a sequence $\{w_k\}\in S_g$ such that $w_k(z)\to g(z)$. Indeed, since $u \ge w_k$ pointwise, we would have $\liminf_{x\to z} u(x) \ge w_k(z)$ for each k. The existence of such a sequence $\{w_k\}$ means, in a certain sense, that the domain Ω is able to support a sufficiently rich family of subharmonic functions. In a similar fashion, to rule out the second scenario, we need to have a sufficiently rich family of "superharmonic" functions, and as "superharmonic" functions are simply the negatives of subharmonic functions, it turns out that both scenarios can be handled by the same method. We start by introducing the concept of a barrier.

Definition 9. A function $\varphi \in \mathfrak{Sub}(\Omega) \cap \mathscr{C}(\bar{\Omega})$ is called a *barrier for* Ω *at* $z \in \partial\Omega$ if

- $\varphi(z) = 0$,
- $\varphi < 0$ on $\partial\Omega \setminus \{z\}$.

We call the boundary point $z \in \partial\Omega$ *regular* if there is a barrier for Ω at $z \in \partial\Omega$.

Lemma 10. *Let $z \in \partial\Omega$ be a regular point, and let g be continuous at z. Then for any given $\varepsilon > 0$, there exists $w \in S_g$ such that $w(z) \geq g(z) - \varepsilon$.*

Proof. Let $\varepsilon > 0$, and let φ be a barrier at z. Then there exists $\delta > 0$ such that $|g(x) - g(z)| < \varepsilon$ for $x \in \partial\Omega \cap B_\delta(z)$. Choose $M > 0$ so large that $M\varphi(x) + 2\|g\|_\infty < 0$ for $x \in \partial\Omega \setminus B_\delta(z)$, and consider the function $w = M\varphi + g(z) - \varepsilon$. Obviously, $w \in \mathfrak{Sub}(\Omega) \cap \mathscr{C}(\bar{\Omega})$ and $w(z) = g(z) - \varepsilon$. Moreover, we have

$$M\varphi(x) + g(z) - \varepsilon < M\varphi(x) + g(x) \leq g(x), \qquad x \in \partial\Omega \cap B_\delta(z), \tag{24}$$

and

$$M\varphi(x) + g(z) - \varepsilon < -2\|g\|_\infty + g(z) \leq g(x), \qquad x \in \partial\Omega \setminus B_\delta(z), \tag{25}$$

which imply that $w \in S_g$. □

Exercise 11. Show that if the Dirichlet problem in Ω is solvable for all boundary conditions $g \in \mathscr{C}(\partial\Omega)$, then each $z \in \partial\Omega$ is a regular point.

The following theorem implies the converse to the preceding exercise: If all boundary points are regular, then the Dirichlet problem is solvable for any $g \in \mathscr{C}(\partial\Omega)$.

Theorem 12 (Perron 1923)**.** *Assume that $z \in \partial\Omega$ is a regular point, and that g is continuous at z. Then we have $u(x) \to g(z)$ as $\Omega \ni x \to z$.*

Proof. By Lemma 10, for any $\varepsilon > 0$ there exists $w \in S_g$ such that $w(z) \geq g(z) - \varepsilon$. By definition, we have $u \geq w$ in Ω. This shows that

$$\liminf_{\Omega \ni x \to z} u(x) \geq g(z) - \varepsilon, \tag{26}$$

and as $\varepsilon > 0$ was arbitrary, the same relation is true with $\varepsilon = 0$. On the other hand, again by Lemma 10, for any $\varepsilon > 0$ there exists $w \in S_{-g}$ such that $w(z) \geq -g(z) - \varepsilon$. Let $v \in S_g$. Then $v + w \in \mathfrak{Sub}(\Omega) \cap \mathscr{C}(\bar{\Omega})$ and $v + w \leq 0$ on $\partial\Omega$. This means that $v \leq -w$ in Ω. Since v is an arbitrary element of S_g, the same inequality is true for u, hence

$$\limsup_{\Omega \ni x \to z} u(x) \leq g(z) + \varepsilon, \tag{27}$$

and as $\varepsilon > 0$ was arbitrary, the same relation is true with $\varepsilon = 0$. □

4. Boundary regularity

As we saw in the preceding section, Perron's method reduces the solvability of the Dirichlet problem into the question of boundary regularity. In this section, we will give a couple of simple criteria to check if a boundary point is regular, and allude to more precise results that will be studied later on.

The following is referred to as *Poincaré's criterion* or the *exterior sphere condition.*

Theorem 13 (Poincaré 1887)**.** *Suppose that $B_r(y) \cap \Omega = \varnothing$ and $\overline{B_r(y)} \cap \partial\Omega = \{z\}$, with $r > 0$. Then z is a regular point.*

Proof. For $n \geq 3$, we claim that

$$\varphi(x) = \frac{1}{|x - y|^{n-2}} - \frac{1}{r^{n-2}}, \qquad x \in \bar{\Omega}, \tag{28}$$

is a barrier at z. Indeed, φ is harmonic in $\mathbb{R}^n \setminus \{y\}$, $\varphi(z) = 0$, and $\varphi(x) < 0$ for $x \in \mathbb{R}^n \setminus \overline{B_r(y)}$. For $n = 2$, it is again straightforward to check that

$$\varphi(x) = \log \frac{1}{|x-y|} - \log \frac{1}{r}, \qquad x \in \bar{\Omega}, \tag{29}$$

is a barrier at z. □

Remark 14. In fact, we have the following criterion due to Lebesgue: The point $0 \in \partial\Omega$ is regular if any $x \in \Omega$ near 0 satisfies $x_n < f(|x'|)$, where $x' = (x_1, \ldots, x_{n-1})$ and $f(r) = ar^{1/m}$ for some $a > 0$ and $m > 0$. The case $m = 1$ is known as *Zaremba's criterion* or the *exterior cone condition.*

The following example shows that Lebesgue's criterion is nearly optimal in the sense that the criterion would not be valid if $f(r) = a/\log \frac{1}{r}$.

Example 15 (Lebesgue 1913)**.** Let $I = \{(0, 0, s) : 0 \leq s \leq 1\} \subset \mathbb{R}^3$ and let

$$v(x) = \int_0^1 \frac{s\mathrm{d}s}{|x - p(s)|} \qquad x \in \mathbb{R}^3 \setminus I, \tag{30}$$

where $p(s) = (0, 0, s) \in I$. Note that v is the potential produced by a charge distribution on I, whose density linearly varies from 0 to 1. Consequently, we have $\Delta v = 0$ in $\mathbb{R}^3 \setminus I$, and in particular, $v \in \mathscr{C}^\infty(\mathbb{R}^3 \setminus I)$. It is easy to compute

$$v(x) = |x - p(1)| - |x| + x_3 \log(1 - x_3 + |x - p(1)|) - x_3 \log(-x_3 + |x|). \tag{31}$$

We will be interested in the behaviour of $v(x)$ as $x \to 0$. First of all, since $-x_3 + |x| \geq 2|x_3|$ for $x_3 \leq 0$, if we send $x \to 0$ while keeping $x_3 \leq 0$, then $v(x) \to 1$. To study what happens when $x_3 > 0$, we write

$$v(x) = v_0(x) - x_3 \log\left(|x_1|^2 + |x_2|^2\right), \tag{32}$$

with

$$v_0(x) = |x - p(1)| - |x| + x_3 \log(1 - x_3 + |x - p(1)|) + x_3 \log(x_3 + |x|). \tag{33}$$

The function v_0 is continuous in $\mathbb{R}^3 \setminus \{0, p(1)\}$ with $v_0(x) \to 1$ as $x \to 0$. Moreover, if we send $x \to 0$ in the region $|x_1|^2 + |x_2|^2 \geq |x_3|^n$ with some n, then we still have $v(x) \to 1$. On the other hand, if we send $x \to 0$ along a curve with $|x_1|^2 + |x_2|^2 = e^{-\alpha/x_3}$ for some constant $\alpha > 0$, then we have $v(x) \to 1 + \alpha$. We also note that because of the singularity at $x_1 = x_2 = 0$ of the last term in (32), we see that $v(x) \to +\infty$ as x approaches $I \setminus \{0\}$. Now we define $\Omega = \{x : v(x) < 1 + \alpha\} \cap B_1$ with a sufficiently large $\alpha > 0$. Then although $v(0)$ can be defined so that v is continuous on $\partial\Omega$, it is not possible to extend v to a function in $\mathscr{C}(\bar{\Omega})$.

Next, consider the Dirichlet problem $\Delta u = 0$ in Ω, and $u = v$ on $\partial\Omega$. Let $M = \|u - v\|_{L^\infty(\Omega)}$, and for $\varepsilon > 0$, let $\Omega_\varepsilon = \Omega \setminus \bar{B}_\varepsilon$. Then the function

$$w(x) = \frac{M\varepsilon}{|x|} \pm (u(x) - v(x)), \tag{34}$$

satisfies $\Delta w = 0$ in Ω_ε and $w \geq 0$ on $\partial\Omega_\varepsilon$. By the minimum principle, we have $w \geq 0$ in Ω_ε, which means that

$$|u(x) - v(x)| \leq \frac{M\varepsilon}{|x|}, \qquad x \in \Omega_\varepsilon. \tag{35}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $u = v$. ⊘

If $p \in \Omega$ is an isolated boundary point, i.e., if p is the only boundary point in a neighbourhood of it, then as Zaremba observed, one cannot specify a boundary condition at p because it would be a removable singularity for the harmonic function in the domain. It is shown by Osgood in 1900 that in *2 dimensions*, p is regular if p is contained in a connected component of $\mathbb{R}^2 \setminus \Omega$ that has at least one point other than p. Thus the only situation it leaves undecided is when

p is disconnected from the rest of $\mathbb{R}^2 \setminus \Omega$ yet is an accumulation point of $\mathbb{R}^2 \setminus \Omega$. Note that Osgood's criterion is purely topological.

Theorem 16 (Osgood 1900). *Let $\Omega \subset \mathbb{R}^2$ be open and let $p \in \partial\Omega$ be contained in a component of $\mathbb{R}^2 \setminus \Omega$ which has more than one point (including p). Then p is regular.*

Proof. It will be convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and without loss of generality, to assume that $p = 0$. Let $w \in \mathbb{C}$ be another point so that both p and w are contained in the same connected component of $\mathbb{C} \setminus \Omega$. After a possible scaling, we can assume that $|w| > 1$. Moreover, since regularity is a local property, we can restrict attention to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, that is, we assume that $\Omega \subset \mathbb{D}$. Let $z_0 \in \Omega$ and consider a branch of logarithm near z_0. This branch can be extended to Ω as a single-valued function, for if it were not, there must exist a closed curve in Ω that goes around the origin. However, it is impossible because there is a connected component of $\mathbb{C} \setminus \Omega$ that contains 0 and w. Denoting the constructed branch by log, we claim that $\varphi(z) = \mathrm{Re}(\log z)^{-1}$ is a barrier. Since $\log z$ is a holomorphic function that vanishes nowhere in Ω, we have $\Delta\varphi = 0$ in Ω. Moreover, we have $\varphi(z) \to 0$ as $z \to 0$ and $\varphi < 0$ in Ω because $\mathrm{Re}(\log z) = \log|z|$ and $|z| < 1$ for $z \in \Omega$. This shows that φ is indeed a barrier at 0. □

5. Unbounded domains

In this section, we shall be concerned here with the Dirichlet problem on *unbounded* domains. One important class of such domains is *exterior domains*, that are by definition domains of the form $\Omega = \mathbb{R}^n \setminus K$ with K compact.

We already know that the weak maximum principle does not hold in general for unbounded domains. This causes issues with uniqueness. For example, the function $u(x, y) = \alpha y$ is harmonic in $\mathbb{R}^2$ for any constant $\alpha \in \mathbb{R}$, and hence the homogenous Dirichlet problem in the upper half plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ has infinitely many distinct solutions. As the example suggests, we can in fact restore uniqueness by controlling the behaviour of u "at infinity."

Let $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. We say that the sequence $\{x_j\}$ *converges to* ∞ if $|x_j| \to \infty$ as $j \to \infty$. This defines a topology on $\hat{\mathbb{R}}^n$ and makes it a compact space. For $\Omega \subset \mathbb{R}^n$ a domain, we denote its boundary in $\hat{\mathbb{R}}^n$ by $\hat{\partial}\Omega$. Note that we simply have

$$\hat{\partial}\Omega = \begin{cases} \partial\Omega & \text{if } \Omega \text{ is bounded,} \\ \partial\Omega \cup \{\infty\} & \text{if } \Omega \text{ is unbounded.} \end{cases} \tag{36}$$

The following version of the maximum principle is valid for unbounded domains and for functions that are not necessarily continuous up to the boundary.

Theorem 17 (Maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain, and let $u \in \mathscr{C}(\Omega)$ be a subharmonic function satisfying*

$$\limsup_{\Omega \ni x \to z} u(x) \leq 0 \qquad \text{for each} \quad z \in \hat{\partial}\Omega. \tag{37}$$

Then we have $u \leq 0$ in Ω.

Proof. The bounded open sets $K_j = B_j \cap \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 1/j\}$ for $j = 1, 2, \ldots$, satisfy $\bigcup_j K_j = \Omega$ and $K_1 \subset K_2 \subset \ldots \subset \Omega$. We claim that

$$\limsup_{j\to\infty} \sup_{x\in\partial K_j} u(x) \leq 0. \tag{38}$$

If this claim is true, then we would have $\sup_{\partial K_j} u \leq \varepsilon_j$ with some (positive) sequence $\{\varepsilon_j\}$ satisfying $\varepsilon_j \to 0$ as $j \to \infty$, and $\sup_{K_j} u \leq \sup_{\partial K_j} u$ by the maximum principle for bounded open sets. Since for any given $x \in \Omega$, we have $x \in K_j$ and hence $u(x) \leq \varepsilon_j$ for all large j, the proof would be complete.

To prove the claim, suppose that (38) is not true. This means that there exists some $\varepsilon > 0$, such that $\sup_{\partial K_j} u \geq \varepsilon$ for infinitely many j. By removing some elements from the collection $\{K_j\}$ if necessary, we can in fact assume that $\sup_{\partial K_j} u \geq \varepsilon$ for all j. Hence there is a sequence $x_j \in \partial K_j$ with $u(x_j) \geq \varepsilon/2$ for all j. As $\hat{\mathbb{R}}^n$ is compact, passing to a subsequence, we can guarantee that $x_j \to z$ for some $z \in \hat{\mathbb{R}}^n$. If $z \neq \infty$, then $\{x_j\}$ is bounded, and hence $\operatorname{dist}(x_j, \partial\Omega) \to 0$ as $j \to \infty$. This means that either $z = \infty$ or $z \in \partial\Omega$, which contradicts the initial hypothesis (37). □

Corollary 18. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $u \in \mathscr{C}^2(\Omega)$ be a harmonic function satisfying*

$$\lim_{\Omega \ni x \to z} u(x) = 0 \qquad \text{for each} \quad z \in \hat{\partial}\Omega. \tag{39}$$

Then we have $u = 0$ in Ω.

Now we prove a generalization of Theorem 8.

Theorem 19. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\mathscr{P} \subset \mathfrak{Sub}(\Omega)$ be a family of subharmonic functions in Ω, satisfying the following conditions.*

- *$\max\{u_1, u_2\} \in \mathscr{P}$ whenever $u_1, u_2 \in \mathscr{P}$.*
- *$u_B \in \mathscr{P}$ whenever $u \in \mathscr{P}$ and B is an open ball with $\bar{B} \subset \Omega$, where $u_B \in \mathscr{C}(\Omega)$ denotes the function harmonic in B which agrees with u in $\Omega \setminus B$.*

Then the Perron solution $u : \Omega \to \mathbb{R} \cup \{\infty\}$, defined by

$$u(x) = \sup_{v \in \mathscr{P}} v(x), \qquad x \in \Omega, \tag{40}$$

satisfies either $u \equiv \infty$ in Ω, or $u \in \mathscr{C}^2(\Omega)$ and $\Delta u = 0$ in Ω.

Proof. This proof is a slight generalization of the proof of Theorem 8. Let $B_r(x)$ be a nonempty open ball whose closure is in Ω, and let $\{u_k\} \subset \mathscr{P}$ be a sequence satisfying $u_k(x) \to u(x)$ as $k \to \infty$. Without loss of generality, we can assume that the sequence is nondecreasing, e.g., by replacing u_k by $\max\{u_1, \ldots, u_k\}$. For each k, let $\bar{u}_k \in \mathscr{C}(\Omega)$ be the function harmonic in $B_r(x)$ which agrees with u_k in $\Omega \setminus B_r(x)$. We have $u_k \leq \bar{u}_k$, and $\bar{u}_k \in \mathscr{P}$ hence $\bar{u}_k(x) \leq u(x)$, so $\bar{u}_k(x) \to u(x)$ as well. The sequence $\{\bar{u}_k\}$ is also nondecreasing, so by the Harnack principle, one of the following two possibilities must hold.

- $\bar{u}_k \to \infty$ everywhere in $B_r(x)$. In this case, we would have $u \equiv \infty$ in $B_r(x)$. Therefore, the set $\Sigma = \{z \in \Omega : u(z) = \infty\}$ is open.
- There exists a harmonic function $\bar{u}$ in $B_r(x)$ such that $\bar{u}_k \to \bar{u}$ locally uniformly in $B_r(x)$. In particular, we have $\bar{u}(x) = u(x)$. As in the proof of Theorem 8, one can show that $u = \bar{u}$ in $B_r(x)$, and hence u is harmonic in Ω. This means that the set $\Omega \setminus \Sigma$ is also open.

Since Ω is connected, we conclude that either $\Sigma = \Omega$ or $\Sigma = \varnothing$. □

Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $g : \hat{\partial}\Omega \to \mathbb{R}$ be a bounded function. Then we set

$$\mathscr{P}_g = \{v \in \mathfrak{Sub}(\Omega) : \limsup_{\Omega \ni x \to z} u(x) \leq g(z) \text{ for each } z \in \hat{\partial}\Omega\}, \tag{41}$$

and define the *Perron solution* $u : \Omega \to \mathbb{R} \cup \{\infty\}$ by

$$u(x) = \sup_{v \in \mathscr{P}_g} v(x), \qquad x \in \Omega. \tag{42}$$

The family $\mathscr{P}_g$ satisfies the hypothesis of Theorem 19, and hence either $u \equiv \infty$ or u is harmonic in Ω. Moreover, as g is bounded, $\mathscr{P}_g$ is nonempty, and any $v \in \mathscr{P}_g$ satisfies $v \leq M$, where M is an upper bound on g. Therefore, u is finite-valued, and hence must be harmonic in Ω.

Exercise 20. In the above setting, let $w \in \mathscr{C}^2(\Omega)$ be a function satisfying $\Delta w = 0$ in Ω, and $w(x) \to g(z)$ as $\Omega \ni x \to z$ for any $z \in \hat{\partial}\Omega$. Then show that $w = u$.

To deal with boundary conditions, we generalize the notion of barriers to unbounded domains.

Definition 21. A function $\varphi \in \mathfrak{Sub}(\Omega)$ is called a *barrier for* Ω *at* $z \in \hat{\partial}\Omega$ if

- $\varphi(x) \to 0$ as $\Omega \ni x \to z$,
- $\sup\limits_{\Omega \setminus B_\delta(z)} \varphi < 0$ for each $\delta > 0$.

We call the boundary point $z \in \hat{\partial}\Omega$ *regular* if there is a barrier for Ω at z.

Remark 22. Definition 21 contains Definition 9 as a special case, and hence in particular Poincaré's and Osgood's criteria are valid for $z \in \partial\Omega$, that is, if $z \neq \infty$. In dimensions $n \geq 3$, the point $z = \infty$ is *always regular*, as one can check that $\varphi(x) = \max\{-1, -|x|^{2-n}\}$ is a barrier at ∞ even for $\Omega = \mathbb{R}^n$. On the other hand, since there is no subharmonic function in $\mathbb{R}^2$ that is bounded from above, the point ∞ is *not* regular (or *irregular*) for $\mathbb{R}^2$. Moreover, the following generalized version of Osgood's criterion is true: $z \in \hat{\partial}\Omega$ for a domain $\Omega \subset \mathbb{R}^2$ is regular if z is contained in a connected component of $\hat{\mathbb{R}}^2 \setminus \Omega$ which has more than one point (including z).

Lemma 23. *Let $z \in \hat{\partial}\Omega$ be a regular point, and let g be continuous at z. Then for any given $\varepsilon > 0$, there exists $w \in \mathscr{P}_g$ such that $\liminf\limits_{\Omega \ni x \to z} w(x) \geq g(z) - \varepsilon$.*

Proof. Let $\varepsilon > 0$, and let φ be a barrier at z. Then there exists $\delta > 0$ such that $|g(y) - g(z)| < \varepsilon$ for $y \in \hat{\partial}\Omega \cap B_\delta(z)$. Here if $z = \infty$, the ball $B_\delta(z)$ is understood to be $\hat{\mathbb{R}}^n \setminus \bar{B}_{1/\delta}(0)$. Choose $M > 0$ so large that $M\varphi(x) + 2\|g\|_\infty < 0$ for $x \in \Omega \setminus B_\delta(z)$, and consider the function $w = M\varphi + g(z) - \varepsilon$. Obviously, $w \in \mathfrak{Sub}(\Omega)$ and $w(x) \to g(z) - \varepsilon$ as $\Omega \ni x \to z$. Moreover, for $x \in \Omega$ and $y \in \hat{\partial}\Omega \cap B_\delta(z)$ we have

$$M\varphi(x) + g(z) - \varepsilon < M\varphi(x) + g(y) \leq g(y), \tag{43}$$

and for $x \in \Omega \setminus B_\delta(z)$ and $y \in \hat{\partial}\Omega \setminus B_\delta(z)$ we have

$$M\varphi(x) + g(z) - \varepsilon < -2\|g\|_\infty + g(z) \leq g(y), \tag{44}$$

which imply that $w \in \mathscr{P}_g$. □

Theorem 24. *Assume that $z \in \hat{\partial}\Omega$ is a regular point, and that g is continuous at z. Then we have $u(x) \to g(z)$ as $\Omega \ni x \to z$.*

Proof. By Lemma 23, for any $\varepsilon > 0$ there exists $w \in \mathscr{P}_g$ such that $\liminf\limits_{\Omega \ni x \to z} w(x) \geq g(z) - \varepsilon$. By definition, we have $u \geq w$ in Ω. This shows that

$$\liminf_{\Omega \ni x \to z} u(x) \geq g(z) - \varepsilon, \tag{45}$$

and as $\varepsilon > 0$ was arbitrary, the same relation is true with $\varepsilon = 0$. On the other hand, again by Lemma 23, for any $\varepsilon > 0$ there exists $w \in \mathscr{P}_{-g}$ such that $\liminf\limits_{\Omega \ni x \to z} w(x) \geq -g(z) - \varepsilon$. Let $v \in \mathscr{P}_g$. Then we have $v + w \in \mathscr{P}_0$, and so the maximum principle yields $v \leq -w$ in Ω. Since v is an arbitrary element of $\mathscr{P}_g$, the same inequality is true for u. This gives

$$\limsup_{\Omega \ni x \to z} u(x) \leq g(z) + \varepsilon, \tag{46}$$

and as $\varepsilon > 0$ was arbitrary, the same relation is true with $\varepsilon = 0$. □

6. Problems and exercises

1. Let Ω be a bounded domain, and let $\{u_j\} \subset \mathscr{C}(\bar{\Omega})$ be a sequence of harmonic functions in Ω. Show that if $\{u_j\}$ converges uniformly on $\partial\Omega$, then it converges uniformly in $\bar{\Omega}$, to a function that is harmonic in Ω.

2. Let Ω be an open set, and let $\{u_j\}$ be a sequence of harmonic functions in Ω, such that $u_j \to u$ in $L^1_{\text{loc}}(\Omega)$ for some $u \in L^1_{\text{loc}}(\Omega)$. This means that for any compact set $K \subset \Omega$, we have $\|u_j - u\|_{L^1(K)} \to 0$ as $j \to \infty$. Show that there is $w \in \mathscr{C}^2(\Omega)$ with $\Delta w = 0$ in Ω such that $u = w$ almost everywhere in Ω.

3. If $z \in \partial\Omega$ is regular, and if Ω' is a domain that coincides with Ω in a neighbourhood of z (hence in particular $z \in \partial\Omega'$), then can you conclude that z is also regular as a point of $\partial\Omega'$? In other words, is regularity of a boundary point a local property?

4. Show that Green's function exists for the domain Ω if each point of $\partial\Omega$ is regular.

5. Complete the proof of Theorem 19.

6. In $\mathbb{R}^n$, define the *Kelvin transform* for functions by

$$(Ku)(x) = |x|^{2-n} u(x/|x|^2).$$

Show that if u is harmonic in $\Omega \subset \mathbb{R}^n$, then Ku is harmonic in

$$\Omega' = \{x/|x|^2 : x \in \Omega \setminus \{0\}\}.$$

What is Ω' if $\Omega = \{x \in \mathbb{R}^n : x_n > 1\}$? Show that $K^{-1} = K$, i.e., that K is an involution.

7 (Poincaré 1887)**.** In this exercise, we will implement Poincaré's *method of sweeping out* (*méthode de balayage*) to solve the Dirichlet problem. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $g \in \mathscr{C}(\Omega)$. Suppose that $u_0 \in \mathscr{C}(\bar{\Omega})$ is a function subharmonic in Ω and $u_0 = g$ on $\partial\Omega$. The idea is to iteratively improve the initial approximation u_0 towards a harmonic function by solving the Dirichlet problem on a suitable sequence of balls.

(a) Show that there exist countably many open balls B_k such that $\Omega = \bigcup_k B_k$.

(b) Consider the sequence $B_1, B_2, B_1, B_2, B_3, B_1, \ldots$, so that each B_k is occurring infinitely many times, and let us reuse the notation B_k to denote the k-th member of this sequence. Then we define the functions $u_1, u_2, \ldots \in \mathscr{C}(\bar{\Omega})$ by the following recursive procedure: For $k = 1, 2, \ldots$, put $u_k = u_{k-1}$ in $\Omega \setminus B_k$, and let u_k be the solution of $\Delta u_k = 0$ in B_k with the boundary condition $u_{k-1}|_{\partial B_k}$. Prove that $u_k \to u$ locally uniformly in Ω, for some $u \in \mathscr{C}^\infty(\Omega)$ that is harmonic in Ω.

(c) Show that if there exists $v \in \mathscr{C}(\bar{\Omega})$ satisfying $\Delta v = 0$ in Ω and $v = g$ on $\partial\Omega$, then indeed $u = v$, where u is the function we constructed in (b). So if there exists a solution, then our method would produce the same solution. However, we want to demonstrate existence without any prior assumption on existence.

(d) Prove that if there exists a barrier at $z \in \partial\Omega$, then $u(x) \to g(z)$ as $\Omega \ni x \to z$, where u is the function we constructed in (b).

(e) Assuming that all boundary points are regular, this procedure reduces the Dirichlet problem into the problem of constructing a subharmonic function u_0 with $u_0|_{\partial\Omega} = g$. Instead of constructing such u_0 for the given g directly, let us approximate g by functions for which such a construction is simpler. Show that if $\{v_j\} \subset \mathscr{C}(\bar{\Omega})$ is a sequence with $\Delta v_j = 0$ in Ω and $v_j \to g$ uniformly on $\partial\Omega$, then there exists a function $u \in \mathscr{C}(\bar{\Omega})$ satisfying $\Delta u = 0$ in Ω and $u = g$ on $\partial\Omega$.

(f) Show that any polynomial can be written as the difference of two subharmonic functions in Ω. Hence it suffices to extend g into a continuos function on $\bar{\Omega}$, and approximate the resulting function by polynomials (explain why). State what standard results we need in order to realize this.