THE DIRICHLET PROBLEM AS A MINIMIZATION PROBLEM

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Abstract. The classical Dirichlet principle is made rigorous, through the introduction of weak and strong derivatives, and Sobolev spaces.

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1. The Dirichlet energy

In this section, we will be introduced to the problem of minimizing the *Dirichlet energy*

$$E(u) = \int_\Omega |\nabla u|^2, \tag{1}$$

subject to the boundary condition $u|_{\partial\Omega} = g$. Recall from the preceding chapter that this approach to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2}$$

was originally suggested around 1847 by William Thomson and Gustav Lejeune-Dirichlet. We start with some simple observations.

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in \mathscr{C}^2(\Omega)$ satisfy $E(u) < \infty$.*

- *If $\Delta u = 0$ in Ω, then $E(u + v) > E(u)$ for all nontrivial $v \in \mathscr{C}^1_c(\Omega)$.*
- *Conversely, if $E(u + v) \geq E(u)$ for all $v \in \mathscr{C}^1_c(\Omega)$, then $\Delta u = 0$ in Ω.*

Proof. Let $v \in \mathscr{D}(\Omega)$ and let $\varepsilon \in \mathbb{R}$. Then we have

$$E(u + \varepsilon v) = E(u) + 2\varepsilon \int_\Omega \nabla u \cdot \nabla v + \varepsilon^2 E(v) = E(u) - 2\varepsilon \int_\Omega v \Delta u + \varepsilon^2 E(v), \tag{3}$$

by Green's first identity and the fact that $\operatorname{supp} v$ is compact. The first assertion of the lemma follows by putting $\Delta u = 0$ and $\varepsilon = 1$. For the second assertion, note that

$$2\varepsilon \int_\Omega v \Delta u = E(u) - E(u + \varepsilon v) + \varepsilon^2 E(v) \leq \varepsilon^2 E(v), \tag{4}$$

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for all $\varepsilon \in \mathbb{R}$, implying that

$$2\Big|\int_\Omega v\Delta u\Big| \leq |\varepsilon| E(v), \qquad \text{and so} \qquad \int_\Omega v\Delta u = 0. \tag{5}$$

Since v is arbitrary and Δu is continuous, we infer that $\Delta u = 0$ in Ω. □

The second assertion of the preceding lemma tells us that in order to establish existence of a solution to the Dirichlet problem (2), it suffices to show that E has a minimizer in

$$\mathscr{A}_0 = \{u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}(\bar{\Omega}) : u|_{\partial\Omega} = g\}. \tag{6}$$

In order to obtain a minimizer, one would start with a sequence $\{u_k\} \subset \mathscr{A}_0$ satisfying

$$E(u_k) \to \mu := \inf_{v\in\mathscr{A}_0} E(v) \qquad \text{as} \quad k \to \infty, \tag{7}$$

and then try to show that this sequence (or some subsequence of it) converges to an element $u \in \mathscr{A}_0$ with $E(u) = \mu$. Such a sequence is called a *minimizing sequence*. The difficulty with this plan is that although one can easily establish the existence of some function u such that $u_k \to u$ in a certain sense, the topology in which the convergence $u_k \to u$ occurs is so weak that we cannot imply the membership $u \in \mathscr{A}_0$ from the convergence alone. Initially, e.g., in the works of David Hilbert and Richard Courant, this difficulty was overcame by modifying the sequence $\{u_k\}$ without loosing its minimizing property, so as to be able to say more about the properties of the limit u. However, it was later realized that the following modular approach is more natural and often better suited for generalization.

- First, we show that E has a minimizer in a class that contains $\mathscr{A}_0$ as a subset.
- Then we show that the minimizer we obtained is in fact in $\mathscr{A}_0$.

The division of labor described here, that separates *existence* questions from *regularity* questions, has became the basic philosophy of calculus of variations. Already in 1900, Hilbert proposed existence and regularity questions (for minimization of more general energies) as two individual problems in his famous list.

In a few sections that follow, we will carry out this program for the Dirichlet energy. Before setting up the problem, let us look at a counterexample due to Jacques Hadamard, which is a variation of Friedrich Prym's example from 1871.

Example 2 (Hadamard 1906)**.** Let $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$, and let $u : \mathbb{D} \to \mathbb{R}$ be given in polar coordinates by

$$u(r,\theta) = \sum_{n=1}^{\infty} n^{-2} r^{n!} \sin(n!\theta). \tag{8}$$

It is easy to check that each term of the series is harmonic, and the series converges absolutely uniformly in $\bar{\mathbb{D}}$. Hence u is harmonic in $\mathbb{D}$ and continuous in $\bar{\mathbb{D}}$. On the other hand, we have

$$E(u) = \int_{\mathbb{D}} |\nabla u|^2 \geq \int_0^{2\pi}\int_0^\rho |\partial_r u(r,\theta)|^2 r\mathrm{d}r\mathrm{d}\theta = \sum_{n=1}^{\infty} \frac{\pi n!}{2n^4}\rho^{2n!} \geq \sum_{n=1}^{m} \frac{\pi n!}{2n^4}\rho^{2n!}, \tag{9}$$

for any $\rho < 1$ and any integer m. This implies that $E(u) = \infty$. To conclude, there exists a Dirichlet datum $g \in C(\partial\mathbb{D})$ for which the Dirichlet problem is perfectly solvable, but the solution cannot be obtained by minimizing the Dirichlet energy. There is no full equivalence between the Dirichlet problem and the minimization problem.

We are now ready to enter the full mathematical set up of the problem. With $\Omega \subset \mathbb{R}^n$ a domain (not necessarily bounded) and $g \in \mathscr{C}(\partial\Omega)$, we would like to minimize E over a class of functions u satisfying the boundary condition $u|_{\partial\Omega} = g$. However, in view of Hadamard's example, we want to make sure that there is at least one function u such that $u|_{\partial\Omega} = g$ and that $E(u) < \infty$. We will implement it by assuming from the beginning that $g \in \mathscr{C}^1(\Omega) \cap \mathscr{C}(\bar{\Omega})$ and that $E(g) < \infty$, so that the boundary condition now takes the form $u|_{\partial\Omega} = g|_{\partial\Omega}$. Furthermore, we introduce the space

$$\mathscr{C}_0^1(\Omega) = \{u \in \mathscr{C}^1(\Omega) \cap \mathscr{C}(\bar{\Omega}) : u|_{\partial\Omega} = 0\}, \tag{10}$$

and let

$$\mathscr{A} = \{u \in \mathscr{C}^1(\Omega) \cap \mathscr{C}(\bar{\Omega}) : u - g \in \mathscr{C}_0^1\}. \tag{11}$$

An alternative, perhaps more standard way of defining the admissible set $\mathscr{A}$ would be to require $u - g \in \mathscr{D}(\Omega)$, which may appear a bit strange as it is not *a priori* clear why we need to impose conditions on the derivatives of $u - g$ near $\partial\Omega$. However, it will turn out that both choices lead to the same outcome. We remark that the differentiability condition is now $\mathscr{C}^1$ in (11) as opposed to $\mathscr{C}^2$ in (6), because the Dirichlet energy makes perfect sense for $\mathscr{C}^1$ functions. This is not very relevant, because as we shall see, we will eventually be using an even larger class of functions.

The next step is to consider a *minimizing sequence*. Let

$$\mu = \inf_{v \in \mathscr{A}} E(v). \tag{12}$$

It is obvious that $0 \leq \mu < \infty$, because $g \in \mathscr{A}$ and $E(g) < \infty$. By definition of infimum, there exists a sequence $\{u_k\} \subset \mathscr{A}$ satisfying

$$E(u_k) \to \mu \qquad \text{as} \quad k \to \infty. \tag{13}$$

We will see that there is a natural topology associated to the energy E in which we have the convergence $u_k \to u$ to some function u. However, this topology will not be very strong, as the following example illustrates.

Example 3 (Courant)**.** Consider the Dirichlet problem on the unit disk $\mathbb{D} \subset \mathbb{R}^2$ with the homogeneous Dirichlet boundary condition. The solution is $u \equiv 0$, which also minimizes the Dirichlet energy. For $k \in \mathbb{N}$, let

$$u_k(r, \theta) = \begin{cases} k a_k & \text{for } r < e^{-2k}, \\ -a_k(k + \log r) & \text{for } e^{-2k} < r < e^{-k}, \\ 0 & \text{for } e^{-k} < r < 1, \end{cases} \tag{14}$$

given in polar coordinates. These are continuous, piecewise smooth functions with

$$E(u_k) = 2\pi \int_0^1 |\partial_r u_k|^2 \, r \mathrm{d}r = 2\pi a_k^2 \log r \Big|_{e^{-2k}}^{e^{-k}} = 2\pi k a_k^2. \tag{15}$$

Upon choosing $a_k = k^{-2/3}$, we can ensure that $\{u_k\}$ is a minimizing sequence. However, $u_k(0) = k a_k = k^{1/3}$ diverges as $k \to \infty$. In any case, observe that u_k converges to $u \equiv 0$ in some averaged sense.

Exercise 4. In the context of the preceding example, construct a minimizing sequence of piecewise smooth functions satisfying the homogeneous boundary condition, which diverges in a set that is dense in $\mathbb{D}$. Show that this sequence converges to $u \equiv 0$ in L^2.

In order to illustrate the main ideas clearly, before dealing with the Dirichlet energy (1), we would like to consider the problem of minimizing the modified energy

$$E_*(u) = \int_\Omega \left(|\nabla u|^2 + |u|^2 \right). \tag{16}$$

The admissible set $\mathscr{A}$ will stay the same, as in (11), and we will assume that

$$\mu_* = \inf_{v \in \mathscr{A}} E_*(v) < \infty. \tag{17}$$

If Ω is bounded, $\mu_* < \infty$ if and only if $\mu < \infty$, because the second term under the integral in (16) is integrable for any $u \in \mathscr{A}$. One can also show that minimizing E_* corresponds to the boundary value problem $\Delta u = u$ in Ω and $u = g$ on $\partial\Omega$.

Exercise 5. Establish an analogue of Lemma 1 for the modified energy E_*. In particular, show that if $u \in \mathscr{C}^2(\Omega)$ satisfies $E_*(u+v) \geq E_*(u)$ for all $v \in \mathscr{C}^1_c(\Omega)$, then $\Delta u = u$ in Ω.

Pick a minimizing sequence for E_*, i.e., let $\{u_k\} \subset \mathscr{A}$ be such that

$$E_*(u_k) \to \mu_* \qquad \text{as} \quad k \to \infty. \tag{18}$$

Note that $E(u) = \langle u, u\rangle_*$, where $\langle \cdot, \cdot \rangle_*$ is the symmetric bilinear form given by

$$\langle u, v\rangle_* = \int_\Omega (\nabla u \cdot \nabla v + uv)\,. \tag{19}$$

Any symmetric bilinear form satisfies the *parallelogram law*:

$$\langle u-v, u-v\rangle_* + \langle u+v, u+v\rangle_* = 2\langle u, u\rangle_* + 2\langle v, v\rangle_*, \tag{20}$$

which reveals that

$$E_*(u_j - u_k) = 2E_*(u_j) + 2E_*(u_k) - 4E_*\big(\frac{u_j+u_k}{2}\big) \leq 2E_*(u_j) + 2E_*(u_k) - 4\mu_*, \tag{21}$$

where the inequality is because of the fact that $\frac{u_j+u_k}{2} \in \mathscr{A}$. Since $\{u_j\}$ is a minimizing sequence, we have $E_*(u_j - u_k) \to 0$ as $j, k \to \infty$. We would have said that $\{u_j\}$ is a Cauchy sequence, for instance, if there was some norm of $u_j - u_k$, rather than $E_*(u_j - u_k)$, that is going to 0. As it turns out, this is indeed true: The quantity $\|\cdot\|_{H^1(\Omega)} = \sqrt{E_*}$, that is, $\|\cdot\|_{H^1(\Omega)}$ given by

$$\|u\|^2_{H^1(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} = \int_\Omega \left(|\nabla u|^2 + |u|^2\right), \tag{22}$$

is a norm for functions in

$$\tilde{\mathscr{C}}^1(\Omega) = \{u \in \mathscr{C}^1(\Omega) : \|u\|_{H^1(\Omega)} < \infty\}. \tag{23}$$

The space $\tilde{\mathscr{C}}^1(\Omega)$ is a proper subset of $\mathscr{C}^1(\Omega)$, for instance, by Hadamard's example, although much simpler examples can be constructed that exploit growth, rather than oscillation, near the boundary of Ω. To conclude, (21) implies that the sequence $\{u_j\}$ is Cauchy with respect to this new norm:

$$\|u_j - u_k\|_{H^1(\Omega)} \to 0, \qquad \text{as} \quad j, k \to \infty. \tag{24}$$

The reader must have recognized why we modified the Dirichlet energy: It is precisely to make $\sqrt{E_*}$ a norm. For the Dirichlet energy, $\sqrt{E}$ is still a norm for functions in $\mathscr{C}^1(\Omega)$ satisfying the homogeneous boundary condition, but to show this one needs a bit more machinery, in particular the Friedrichs inequality (Theorem 19 below). Note that since $u_j - u_k$ satisfies the homogeneous boundary condition, this would have been sufficient for us. The Dirichlet energy E will be taken up after the treatment of E_* which is a bit simpler.

2. Strong derivatives and weak solutions

Returning back to minimizing E_*, we have shown that any minimizing sequence is a Cauchy sequence with respect to the norm $\|\cdot\|_{H^1(\Omega)}$. Now, if $\tilde{\mathscr{C}}^1(\Omega)$ was complete with respect to the norm $\|\cdot\|_{H^1(\Omega)}$, there would have been $u \in \tilde{\mathscr{C}}^1(\Omega)$ such that $\|u_j - u\|_{H^1(\Omega)} \to 0$. However, it is a fact of life that $\tilde{\mathscr{C}}^1(\Omega)$ is *not complete* with respect to the norm $\|\cdot\|_{H^1(\Omega)}$.

Exercise 6. Let $v(x) = \log\log(2/|x|)$ and let $v_k \in \mathscr{C}(\mathbb{D})$ be defined by $v_k(x) = \min\{k, v(x)\}$. Show that the norms $\|v_k\|_{H^1(\mathbb{D})}$ are uniformly bounded. Exhibit a sequence $\{u_k\} \subset \tilde{\mathscr{C}}^1(\mathbb{D})$ that is Cauchy with respect to the norm $\|\cdot\|_{H^1(\mathbb{D})}$, whose limit is not essentially bounded.

If we ignore every aspect of the problem except the fact that $\{u_j\}$ is a Cauchy sequence, the best thing we can do is to consider the *completion* of $\tilde{\mathscr{C}}^1(\Omega)$. What we get in this way is a member of a large family of function spaces called *Sobolev spaces*, named in honour of Sergei Sobolev, who initiated the systematic study of these spaces.

Definition 7. We define the *Sobolev space* $H^1(\Omega)$ as the completion of $\tilde{\mathscr{C}}^1(\Omega)$ with respect to the norm $\|\cdot\|_{H^1(\Omega)}$. In addition, we define $H^1_0(\Omega)$ as the closure of $\mathscr{C}^1_0(\Omega)\cap H^1(\Omega)$ in $H^1(\Omega)$.

If there is no risk of confusion, we will simply write $\|\cdot\|_{H^1}$ omitting from the notation the domain Ω, and call this norm the H^1*-norm*. Note that $\mathscr{C}^1_0(\Omega)\cap H^1(\Omega)$ simply means those elements of $\mathscr{C}^1_0(\Omega)$ with finite H^1-norms. By construction, $H^1_0(\Omega)$ is a closed subspace of $H^1(\Omega)$. For now, $H^1(\Omega)$ is a space whose elements are equivalence classes of Cauchy sequences. We want to identify $H^1(\Omega)$ with a subspace of $L^2(\Omega)$, which would give us a concrete handle on $H^1(\Omega)$. We start with the following observation: If a sequence $\{\phi_k\}\subset\tilde{\mathscr{C}}^1(\Omega)$ is Cauchy with respect to the H^1-norm, then each of the sequences $\{\phi_k\}$ and $\{\partial_i\phi_k\}$, where $i=1,\ldots,n$, is Cauchy in $L^2(\Omega)$. In particular, since $L^2(\Omega)$ is a complete space, there exists a function $u\in L^2(\Omega)$ such that $\phi_k\to u$ in $L^2(\Omega)$ as $k\to\infty$. This defines a map from $H^1(\Omega)$ into $L^2(\Omega)$: It sends the element of $H^1(\Omega)$ represented by the sequence $\{\phi_k\}$ to $u\in L^2(\Omega)$. Let us call this map $J_0: H^1(\Omega)\to L^2(\Omega)$. We will eventually prove that J_0 is injective, identifying $H^1(\Omega)$ as a subspace of $L^2(\Omega)$. For the time being, let us look into the range of J_0. It is clear that a function $u\in L^2(\Omega)$ is in the range of J_0 if and only if there exist a sequence $\{\phi_k\}\subset\tilde{\mathscr{C}}^1(\Omega)$ and functions $v_i\in L^2(\Omega)$ for $i=1,\ldots,n$, such that

$$\phi_k\to u,\qquad\text{and}\qquad \partial_i\phi_k\to v_i,\quad (i=1,\ldots,n),\tag{25}$$

with all convergences taking place in $L^2(\Omega)$. This leads to the concept of strong derivatives, which is based on *approximation*.

Definition 8. For $u,v\in L^2_{\text{loc}}(\Omega)$, we say that $v=\partial_i u$ in the *strong* L^2*-sense*, or that v is a *strong* L^2 *derivative* of u, if for each compact set $K\subset\Omega$, there exists a sequence $\{\phi_k\}\subset\mathscr{C}^1(K)$ such that

$$\phi_k\to u\qquad\text{and}\qquad\partial_i\phi_k\to v\qquad\text{as}\quad k\to\infty,\tag{26}$$

with both convergences taking place in $L^2(K)$.

In particular, if $u\in L^2(\Omega)$ is in the range of J_0, then u is strongly L^2 differentiable.

Example 9. a) Let $u\in\mathscr{C}^1(\Omega)$. Then taking the constant sequence $\phi_k=u$ (for all k) shows that the classical derivative $\partial_i u$ is also a strong L^2 derivative of u.

b) Let $u(x)=|x|$ for $x\in\mathbb{R}$, and let $\phi_k(x)=\sqrt{x^2+\varepsilon^2}$ with $\varepsilon=1/k$. Obviously, we have $\phi_k\in\mathscr{C}^\infty(\mathbb{R})$. From the Maclaurin series of $\sqrt{1+x}$, we get

$$\sqrt{x^2+\varepsilon^2}=|x|\sqrt{1+\frac{\varepsilon^2}{x^2}}=|x|(1+e(x)),\qquad |e(x)|\leq\frac{\varepsilon^2}{x^2},\tag{27}$$

for $\varepsilon<|x|$, and hence

$$\big|\phi_k(x)-|x|\big|=\big|\sqrt{x^2+\varepsilon^2}-|x|\big|\leq\frac{\varepsilon^2}{|x|}\qquad\text{for}\quad |x|>\varepsilon.\tag{28}$$

On the other hand, we have $\phi_k(x)\leq\sqrt{2}\varepsilon$ for $|x|\leq\varepsilon$, so that

$$\int_{-\varepsilon}^{\varepsilon}\big|\phi_k(x)-|x|\big|^2\mathrm{d}x\leq(4\varepsilon^2+2\varepsilon^2)\cdot 2\varepsilon.\tag{29}$$

Together with (28) this implies that $\phi_k\to u$ in $L^2_{\text{loc}}(\mathbb{R})$, because

$$\int_{-a}^{a}\big|\phi_k(x)-|x|\big|^2\mathrm{d}x\leq 12\varepsilon^3+2a\varepsilon,\tag{30}$$

for any $a > 0$. Now we look at

$$\phi_k'(x) = \frac{x}{\sqrt{x^2+\varepsilon^2}}, \tag{31}$$

and would like to show that ϕ_k' converges in $L^2_{\mathrm{loc}}(\mathbb{R})$ to the sign function

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0. \end{cases} \tag{32}$$

This would show that the sign function is a derivative of the absolute value function in the strong L^2-sense. Since ϕ_k' and sign are both odd functions, it suffices to consider only the half line $x > 0$. We observe that $\phi_k'(x) \leq 1$, and that

$$1 - \phi_k'(x) = \frac{\sqrt{x^2+\varepsilon^2} - x}{\sqrt{x^2+\varepsilon^2}} \leq \frac{\varepsilon^2}{x^2}, \qquad \text{for} \quad |x| > \varepsilon, \tag{33}$$

where we have used (28). Then for any fixed $a > 0$ and k large (hence ε small), we compute

$$\int_0^a |\phi_k'(x) - 1|^2 \mathrm{d}x = \int_0^{\sqrt{\varepsilon}} |\phi_k'(x) - 1|^2 \mathrm{d}x + \int_{\sqrt{\varepsilon}}^a |\phi_k'(x) - 1|^2 \mathrm{d}x \leq 4\sqrt{\varepsilon} + a\varepsilon, \tag{34}$$

which confirms the desired convergence.

Proceeding further, for each $i \in \{1, \ldots, n\}$, we can define the map $J_i : H^1(\Omega) \to L^2(\Omega)$ that captures the L^2-limit of $\partial_i \phi_k$ as $k \to \infty$, where $\{\phi_k\}$ is a sequence representing an element of $H^1(\Omega)$. The composite map $J = (J_0, \ldots, J_n) : H^1(\Omega) \to L^2(\Omega)^{n+1}$ is clearly injective, because if $\lim \phi_k = \lim \psi_k$ and $\lim \partial_i \phi_k = \lim \partial_i \psi_k$ for all i, then the mixed sequence $\phi_1, \psi_1, \phi_2, \psi_2, \ldots$ is Cauchy in $H^1(\Omega)$, and so the two sequences $\{\phi_k\}$ and $\{\psi_k\}$ represent the same element of $H^1(\Omega)$. We see that the injectivity of J_0 would follow once we have shown that for any $U \in H^1(\Omega)$, the components $J_1 U, \ldots, J_n U$ are uniquely determined by $J_0 U$ alone. The following result confirms this.

Lemma 10. *Strong derivatives are unique if they exist. Moreover, if $u \in L^2_{\mathrm{loc}}(\Omega)$ is strongly L^2 differentiable, then we have the integration by parts formula*

$$\int_\Omega \varphi \partial_i u = -\int_\Omega u \partial_i \varphi, \qquad \varphi \in \mathscr{C}^1_c(\Omega). \tag{35}$$

Proof. Suppose that both $v, w \in L^2_{\mathrm{loc}}(\Omega)$ are strong L^2 derivative of $u \in L^2_{\mathrm{loc}}(\Omega)$. We want to show that $v = w$ almost everywhere. Let $\varphi \in \mathscr{C}^1_c(\Omega)$, and put $K = \operatorname{supp} \varphi$. Then there is a sequence $\{v_k\} \subset \mathscr{C}^1(K)$ such that $v_k \to u$ and $\partial_i v_k \to v$ as $k \to \infty$, both convergences in $L^2(K)$. From the usual integration by parts, we have

$$\int_\Omega v\varphi = \int_\Omega (v - \partial_i v_k)\varphi + \int_\Omega \varphi \partial_i v_k = \int_\Omega (v - \partial_i v_k)\varphi - \int_\Omega v_k \partial_i \varphi, \tag{36}$$

hence

$$\begin{aligned} \Big|\int_\Omega v\varphi + \int_\Omega u\partial_i\varphi\Big| &\leq \int_\Omega |v - \partial_i v_k||\varphi| + \int_\Omega |u - v_k||\partial_i\varphi| \\ &\leq \|v - \partial_i v_k\|_{L^2(K)} \|\varphi\|_{L^2} + \|u - v_k\|_{L^2(K)} \|\partial_i \varphi\|_{L^2}, \end{aligned} \tag{37}$$

showing that the formula (35) is valid. The same reasoning applies to w, which means that

$$\int_\Omega (v - w)\varphi = \int_\Omega (u - u)\partial_i \varphi = 0. \tag{38}$$

Since $\varphi \in \mathscr{C}^1_c(\Omega)$ is arbitrary, by the du Bois-Reymond lemma (Lemma 22 in §4) we conclude that $v = w$ almost everywhere. □

Remark 11. The heart of the uniqueness argument was the integration by parts formula (35). We will see in §5 that in fact the property (35) characterizes strong derivatives.

In terms of the new concepts we have just defined, we can say that the minimizing sequence $\{u_j\} \subset \mathscr{A}$ converges to some $u \in H^1(\Omega)$. Moreover, from the definition (11), the sequence $\{v_k\}$ defined by $v_k = u_k - g$ is in $\mathscr{C}^1_0(\Omega) \cap H^1(\Omega)$, and it is Cauchy in $H^1(\Omega)$, hence $u - g \in H^1_0(\Omega)$. We emphasize here that the only part of the boundary condition that survives the limit process is $u - g \in H^1_0(\Omega)$, and this must be understood as a generalized form of the Dirichlet boundary condition. The energy E_* is a continuous function on $\tilde{\mathscr{C}}^1(\Omega)$ with respect to the H^1-norm, that can be seen, for instance, from the inequality

$$|E_*(\phi) - E_*(\psi)| \leq \|\phi + \psi\|_{H^1(\Omega)} \|\phi - \psi\|_{H^1(\Omega)}, \qquad \phi, \psi \in \tilde{\mathscr{C}}^1(\Omega). \tag{39}$$

Hence E_* can be extended to a continuous function on $H^1(\Omega)$ in a unique way. Keeping the notation E_* for this extension, we have

$$E_*(u) = E_*(\lim u_j) = \lim E_*(u_j) = \mu_*. \tag{40}$$

We cannot say that u minimizes the energy E_* over $\mathscr{A}$, because we have not ruled out the possibility $u \notin \mathscr{A}$. What we *can* say though is that u minimizes E_* over the set

$$\tilde{\mathscr{A}} = \{g + v : v \in H^1_0(\Omega)\} \supset \mathscr{A}, \tag{41}$$

since $u - g \in H^1_0(\Omega)$ and for any $w \in \tilde{\mathscr{A}}$ there is a sequence $\{w_k\} \subset \mathscr{A}$ converging to w in $H^1(\Omega)$, meaning that

$$E_*(w) = E_*(\lim w_k) = \lim E_*(w_k) \geq \mu_*. \tag{42}$$

Let us now try to derive a differential equation from the minimality of u, as was done in Lemma 1. It is easy to see that $E_*(w)$ can be calculated by the same formula

$$E_*(w) = \int_\Omega \left(|\nabla w|^2 + |w|^2\right), \tag{43}$$

also for $w \in H^1(\Omega)$, with $\nabla w = (\partial_1 w, \ldots, \partial_n w)$ understood in the strong L^2-sense. In light of this, we have

$$E_*(u) \leq E_*(u + \varepsilon v) = E_*(u) + \varepsilon^2 E_*(v) + 2\varepsilon \int_\Omega (\nabla u \cdot \nabla v + uv), \tag{44}$$

for $\varepsilon \in \mathbb{R}$ and $v \in H^1_0(\Omega)$, which then implies that

$$\langle u, v\rangle_{H^1} := \int_\Omega (\nabla u \cdot \nabla v + uv) = 0, \qquad \text{for all} \quad v \in H^1_0(\Omega). \tag{45}$$

We cannot go any further because we cannot quite move the derivatives from v to u in such a low regularity setting (cf. Exercise 5). Until we can prove that u is indeed smooth, we will have to work with (45) as it is.

Definition 12. If $u \in H^1(\Omega)$ satisfies (45), then we say that u solves $\Delta u = u$ in Ω in the *weak sense*, or that u is a *weak solution* of $\Delta u = u$ in Ω. In the same spirit, we call (45) the *weak formulation* of the equation $\Delta u = u$ in Ω.

We have practically proved the following result.

Theorem 13. *Let $g \in H^1(\Omega)$. Then there exists a unique $u \in H^1(\Omega)$ satisfying* (45) *and $u - g \in H^1_0(\Omega)$. In other words, there is a unique weak solution of $\Delta u = u$ with $u - g \in H^1_0(\Omega)$.*

Proof. For uniqueness, let us start as usual by assuming that there exist two such functions $u_1, u_2 \in H^1(\Omega)$. Then $w = u_1 - u_2 \in H^1_0(\Omega)$, and by linearity, we have

$$\int_\Omega (\nabla w \cdot \nabla v + wv) = 0, \tag{46}$$

for all $v \in H^1_0(\Omega)$. Taking $v = w$ gives $\|w\|_{H^1} = 0$, hence $w = 0$.

Existence had already been established, modulo the fact that we now allow $g \in H^1(\Omega)$. For completeness, let us sketch a proof. We define the admissible set $\tilde{\mathscr{A}}$ as in (41), and take a minimizing sequence $\{u_j\} \subset \tilde{\mathscr{A}}$, that is, a sequence satisfying

$$E_*(u_j) \to \mu_* = \inf_{v \in \tilde{\mathscr{A}}} E_*(v). \tag{47}$$

We have $0 \leq \mu_* < \infty$, since $E_*(g) = \|g\|^2_{H^1} < \infty$. The argument (21) shows that $\{u_j\}$ is Cauchy in $H^1(\Omega)$, and hence there is $u \in H^1(\Omega)$ such that $u_j \to u$ in H^1. By continuity of E_*, i.e., the argument (40), we have $E_*(u) = \mu_*$. Moreover, the sequence $\{u_j - g\}$ is Cauchy in $H^1(\Omega)$, and $H^1_0(\Omega)$ is a closed subspace of $H^1(\Omega)$, implying that $u - g \in H^1_0(\Omega)$. Finally, the argument (44) confirms that (45) is satisfied. □

Exercise 14 (Stability)**.** Let $u_1 \in H^1(\Omega)$ and $u_2 \in H^1(\Omega)$ be the weak solutions of $\Delta u = u$ satisfying $u_1 - g_1 \in H^1_0(\Omega)$ and $u_2 - g_2 \in H^1_0(\Omega)$, where $g_1, g_2 \in H^1(\Omega)$. Show that

$$\|u_1 - u_2\|_{H^1} \leq \|g_1 - g_2\|_{H^1}. \tag{48}$$

3. Minimization of the Dirichlet energy

We have proved that the energy E_* attains its minimum over the set $\tilde{\mathscr{A}}$, and that the minimizer is the weak solution to $\Delta u = u$ in Ω. If we can show that u is smooth, then this would imply that $\Delta u = u$ pointwise in Ω. Leaving the smoothness question aside for the moment, now we would like to return to our original goal, that is to minimize the Dirichlet energy E over $\tilde{\mathscr{A}}$. To this end, let us try to imitate and adapt the proof of Theorem 13. Recall that the admissible set $\tilde{\mathscr{A}}$ is defined in (41) with some $g \in H^1(\Omega)$. Let $\{u_j\} \subset \tilde{\mathscr{A}}$ be a minimizing sequence, i.e., let

$$E(u_j) \to \mu = \inf_{v \in \tilde{\mathscr{A}}} E(v). \tag{49}$$

We have $0 \leq \mu < \infty$, since $E(g) = \|\nabla g\|^2_{L^2} < \infty$. Repeating the argument (21), we find that

$$\|\nabla(u_j - u_k)\|_{L^2} \to 0, \qquad \text{as} \quad j, k \to \infty. \tag{50}$$

As $\|\nabla \cdot \|_{L^2}$ is only a part of the H^1-norm, we cannot directly say that the sequence $\{u_j\}$ is Cauchy in H^1. In particular, the fact that $\|\nabla v\|_{L^2} = 0$ would only mean that v is a constant function. However, if we know that $v = 0$ on $\partial\Omega$, then this constant must be 0. This is the intuitive reason behind the *Friedrichs inequality*[1]

$$\|v\|_{H^1} \leq c\|\nabla v\|_{L^2}, \qquad v \in H^1_0(\Omega), \tag{51}$$

where $c > 0$ is a constant. Under the assumption that the Friedrichs inequality is true, from (50) it is immediate that $\{u_j\}$ is Cauchy in H^1, because $u_j - u_k \in H^1_0(\Omega)$. Proceeding as in the proof of Theorem 13, we conclude that $u_j \to u$ in H^1 for some $u \in H^1(\Omega)$ satisfying $E(u) = \mu$ and $u - g \in H^1_0(\Omega)$.

Exercise 15. Show that the function u from the preceding paragraph satisfies

$$\int_\Omega \nabla u \cdot \nabla v = 0, \qquad \text{for all} \quad v \in H^1_0(\Omega). \tag{52}$$

Show also that there is a unique $u \in H^1(\Omega)$ satisfying (52) and $u - g \in H^1_0(\Omega)$.

Definition 16. If $u \in H^1(\Omega)$ satisfies (52), then we say that u solves $\Delta u = 0$ in Ω in the *weak sense*, or that u is a *weak solution* of $\Delta u = 0$ in Ω. We call (52) the *weak formulation* of the equation $\Delta u = 0$ in Ω.

[1]It is sometimes called the *Poincaré inequality*, although the latter term is used more commonly to refer to the same inequality for functions $v \in H^1(\Omega)$ with vanishing mean.

Modulo the proof of Friedrichs' inequality that will follow, we have established the following.

Theorem 17. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $g \in H^1(\Omega)$. Then there exists a unique $u \in H^1(\Omega)$ satisfying $u - g \in H^1_0(\Omega)$ and (52). In other words, there is a unique weak solution of $\Delta u = 0$ with $u - g \in H^1_0(\Omega)$.*

Before proving the Friedrichs inequality, we include here a density result.

Theorem 18. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then $\mathscr{C}^1_c(\Omega)$ is dense in $H^1_0(\Omega)$. In other words, $H^1_0(\Omega)$ could have been defined as the closure of $\mathscr{C}^1_c(\Omega)$ in $H^1(\Omega)$.*

Proof. Recall that $H^1_0(\Omega)$ is defined as the closure of $\mathscr{C}^1_0(\Omega) \cap H^1(\Omega)$ in $H^1(\Omega)$. In order to establish the claimed density, we let $u \in \mathscr{C}^1_0(\Omega) \cap H^1(\Omega)$, and shall construct a sequence in $\mathscr{C}^1_c(\Omega)$, that converges to u in the H^1-norm. To this end, pick a function $\theta \in \mathscr{C}^1(\mathbb{R})$ such that $\theta \equiv 0$ on $[-1, 1]$, and

$$\frac{\theta(t)}{t} \to 1 \qquad \text{and} \qquad \theta'(t) \to 1, \qquad \text{as} \quad |t| \to \infty. \tag{53}$$

Then we define $\theta_k(t) = \frac{\theta(kt)}{k}$ for $t \in \mathbb{R}$, and $u_k(x) = \theta_k(u(x))$ for $x \in \Omega$, where $k \in \mathbb{N}$. Since the set $\{x \in \Omega : |u(x)| \geq \varepsilon\}$ is compact for any $\varepsilon > 0$, and $\theta_k \equiv 0$ on $[-\frac{1}{k}, \frac{1}{k}]$, the support of u_k is compact in Ω. Now for any fixed $x \in \Omega$ with $u(x) \neq 0$, we have

$$u_k(x) = \frac{\theta(ku(x))}{k} \to u(x) \qquad \text{as} \quad k \to \infty, \tag{54}$$

and for those $x \in \Omega$ with $u(x) = 0$, we have $u_k(x) = 0$ anyways, meaning that u_k converges pointwise to u. Moreover, for $u(x) \neq 0$, we have

$$u(x) - u_k(x) = u(x) - \frac{\theta(ku(x))}{k} = u(x)\big(1 - \frac{\theta(ku(x))}{ku(x)}\big), \tag{55}$$

and so

$$|u(x) - u_k(x)| \leq c|u(x)|. \tag{56}$$

Then by the dominated convergence theorem, u_k converges to u in $L^2(\Omega)$.

Next, we need to look at the derivatives. We have

$$\partial_i u_k(x) = \theta_k'(u(x))\, \partial_i u(x) = \theta'(ku(x))\, \partial_i u(x), \tag{57}$$

which means that $\nabla u_k(x) \to \nabla u(x)$ as $k \to \infty$, *except* at those $x \in \Omega$ with $u(x) = 0$ and $\nabla u(x) \neq 0$. However, by the implicit function theorem, this exceptional set is an $n - 1$ dimensional surface, hence has measure zero. Thus ∇u_k converges almost everywhere to ∇u. Furthermore, it follows from (57) that

$$|\partial_i u(x) - \partial_i u_k(x)| \leq |1 - \theta'(ku(x))|\, |\partial_i u(x)| \leq c|\partial_i u(x)|, \tag{58}$$

for almost every $x \in \Omega$, and we get $\partial_i u_k \to \partial_i u$ in $L^2(\Omega)$ by dominated convergence. □

Now we prove the Friedrichs inequality.

Theorem 19 (Friedrichs inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then we have*

$$\|v\|_{L^2} \leq \operatorname{diam}(\Omega) \|\nabla v\|_{L^2}, \qquad \textit{for all} \quad v \in H^1_0(\Omega). \tag{59}$$

Proof. First, we will prove the inequality for $v \in \mathscr{C}^1_c(\Omega)$. Let us extend v by 0 outside Ω so that we have $v \in \mathscr{C}^1_c(\mathbb{R}^n)$. Without loss of generality, assume that $\Omega \subset (0, a)^n$ for some $a > 0$. Then for $x \in \Omega$, we have

$$|v(x)| = \big| \int_0^{x_n} \partial_n v(x', t) \mathrm{d}t \big| \leq \int_0^a |\partial_n v(x', t)| \mathrm{d}t, \tag{60}$$

where $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. The function $t \mapsto |\partial_n v(x', t)|$ is Riemann integrable because it is a continuous function with compact support. Now using the Cauchy-Bunyakowsky-Schwarz inequality and squaring, we get

$$|v(x)|^2 \leq a \int_0^a |\partial_n v(x', t)|^2 \mathrm{d}t, \tag{61}$$

which, upon integrating along $x' = \text{const}$, gives

$$\int_0^a |v(x', t)|^2 \mathrm{d}t \leq a^2 \int_0^a |\partial_n v(x', t)|^2 \mathrm{d}t. \tag{62}$$

Then we integrate over $x' \in (0, a)^{n-1}$, and obtain

$$\int_\Omega |v|^2 = \int_{(0,d)^n} |v|^2 \leq a^2 \int_{(0,a)^n} |\partial_n v|^2 = a^2 \int_\Omega |\partial_n v|^2 \leq a^2 \int_\Omega |\nabla v|^2. \tag{63}$$

This establishes the inequality for $v \in \mathscr{C}^1_c(\Omega)$.

Now let $v \in H^1_0(\Omega)$. Then by Theorem 18 there exists a sequence $\{v_k\} \subset \mathscr{C}^1_c(\Omega)$ such that $v_k \to v$ in $H^1(\Omega)$. The triangle inequality gives

$$\begin{aligned} \|v\|_{L^2} &\leq \|v_k\|_{L^2} + \|v - v_k\|_{L^2} \leq a\|\nabla v_k\|_{L^2} + \|v - v_k\|_{L^2} \\ &\leq a\|\nabla v\|_{L^2} + a\|\nabla v_k - \nabla v\|_{L^2} + \|v - v_k\|_{L^2}, \end{aligned} \tag{64}$$

and since the last two terms can be made arbitrarily small, the claim follows. □

4. Interior regularity: Weyl's lemma

Now that we have established the existence of a minimizer for the Dirichlet energy, in this section, we want to look at how smooth the minimizer is, and if the minimizer satisfies the equation $\Delta u = 0$ in the classical sense. Both questions can be answered simultaneously and affirmatively, as was done by Hermann Weyl in 1940.

Our approach will be to construct a sequence $\{u_j\}$ of harmonic functions such that $u_j \to u$ in L^1_{loc}, which would then establish the desired result since harmonic functions are closed under the convergence in L^1_{loc}. To construct such an approximating sequence, we will employ the technique of *mollifiers* due to Jean Leray, Sergei Sobolev and Kurt Otto Friedrichs, as it is also useful in many other problems. Let $\rho \in \mathscr{D}(B_1)$ where $B_1 \subset \mathbb{R}^n$ is the unit ball, satisfying $\rho \geq 0$ and $\int \rho = 1$. Then we define $\rho_\varepsilon \in \mathscr{D}(B_\varepsilon)$ for $\varepsilon > 0$ by

$$\rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon). \tag{65}$$

It is easy to see that $\int \rho_\varepsilon = 1$. Given $u \in L^1_{\text{loc}}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open, let

$$u_\varepsilon(x) = \int_\Omega \rho_\varepsilon(x - y) u(y) \mathrm{d}y, \qquad x \in \Omega. \tag{66}$$

Note that for each $x \in \Omega$, the integral defining $u_\varepsilon(x)$ makes sense for all sufficiently small $\varepsilon > 0$. The function u_ε could be called a mollified version of u, because it is the outcome of a local averaging process, and as we shall see, u_ε is a smooth function.

Theorem 20. *In this setting, we have the following.*

a) If $u \in \mathscr{C}(\Omega)$, then $u_\varepsilon \to u$ locally uniformly as $\varepsilon \to 0$.

b) If $u \in L^q_{\text{loc}}(\Omega)$ for some $1 \leq q < \infty$, then $u_\varepsilon \to u$ in $L^q_{\text{loc}}(\Omega)$ as $\varepsilon \to 0$.

Proof. a) Making use of the facts $\int \rho_\varepsilon = 1$ and $\rho_\varepsilon \geq 0$, we can write

$$|u(x) - u_\varepsilon(x)| \leq \int_\Omega \rho_\varepsilon(x - y)|u(x) - u(y)| \mathrm{d}y \leq \sup_{y \in B_\varepsilon(x)} |u(x) - u(y)| = \omega(x, \varepsilon), \tag{67}$$

where the last equality defines the function $\omega : K \times (0, \varepsilon_0) \to \mathbb{R}$, with $K \subset \Omega$ an arbitrary compact set and $\varepsilon_0 > 0$ small, depending on K. Since u is continuous, ω is continuous in $K \times (0, \varepsilon_0)$, and moreover ω can be continuously extended to $K \times [0, \varepsilon_0)$ with $\omega(\cdot, 0) = 0$. This shows that $\omega(x, \varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly in $x \in K$, and part a) follows.

b) Let $K \subset \Omega$ and $K' \subset \Omega$ be compact sets, with K contained in the interior of K'. Then the Hölder inequality gives

$$|u_\varepsilon(x)|^q \leq \Big(\int \rho_\varepsilon \Big)^{q-1} \int_{K'} \rho_\varepsilon(x - y) |u(y)|^q \mathrm{d}y, \tag{68}$$

and integrating over $x \in K$, we get

$$\int_K |u_\varepsilon|^q \leq \int_{K'} \Big(\int_K \rho_\varepsilon(x - y) \mathrm{d}x \Big) |u(y)|^q \mathrm{d}y \leq \int_{K'} |u|^q, \tag{69}$$

for small $\varepsilon > 0$. Now let $\delta > 0$ be an arbitrary small number, and let $\phi \in \mathscr{C}(K')$ be such that $\|\phi - u\|_{L^q(K')} < \delta$. The existence of such ϕ is guaranteed by the standard density result, which we recall below in Lemma 21. From the bound we just proved, taking into account the linearity of the mollification process, we have

$$\|\phi_\varepsilon - u_\varepsilon\|_{L^q(K)} \leq \|\phi - u\|_{L^q(K')} < \delta. \tag{70}$$

Finally, we use the triangle inequality to obtain

$$\begin{aligned} \|u_\varepsilon - u\|_{L^q(K)} &\leq \|u_\varepsilon - \phi_\varepsilon\|_{L^q(K)} + \|\phi_\varepsilon - \phi\|_{L^q(K)} + \|\phi - u\|_{L^q(K)} \\ &< \|\phi_\varepsilon - \phi\|_{L^q(K)} + 2\delta \\ &\leq \mathrm{vol}(K)^{1/q} \sup_K |\phi_\varepsilon - \phi| + 2\delta, \end{aligned} \tag{71}$$

which, by part a), implies that $\|u_\varepsilon - u\|_{L^q(K)} < 3\delta$ for all sufficiently small ε, and since $\delta > 0$ is arbitrary, the claim follows. □

We now give a proof of the density result we have used.

Lemma 21. *Let $K \subset \mathbb{R}^n$ be a compact set, and let $1 \leq q < \infty$. Then the space of continuous functions on K is dense in $L^q(K)$.*

Proof. Strictly speaking, an element of $L^q(K)$ is an equivalence class of functions that differ on sets of measure zero. We assume that $g : K \to \mathbb{R}$ is a member of such an equivalence class, and shall prove that for any $\varepsilon > 0$, there is $v \in \mathscr{C}(K)$ such that $\|g - v\|_{L^q(K)} < \varepsilon$. This will suffice since for any other member $\tilde{g}$ of the same class, it holds that $\|g - v\|_{L^q(K)} = \|\tilde{g} - v\|_{L^q(K)}$. By decomposing g into its positive and negative parts, we can assume that g takes only nonnegative values, i.e., that $g : K \to [0, \infty)$. Then for $m \in \mathbb{N}$, we define

$$v_m = \sum_{k=0}^{2^{2m}-1} \frac{k}{2^m} \chi_{A_k} + 2^m \chi_B, \tag{72}$$

where $A_k = \{x \in K : \frac{k}{2^m} < g(x) \leq \frac{k+1}{2^m}\}$ and $B = \{x \in K : g(x) > 2^m\}$. For any set S, the characteristic function χ_S is defined as $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise. By construction, the sequence v_m is nondecreasing, and $v_m \to g$ pointwise. Since g is measurable, the sets A_k and B are also measurable, and so are the functions v_m. Moreover, we have

$$|g - v_m|^q \leq 2^{q-1}|g|^q + 2^{q-1}|v_m|^q \leq 2^q |g|^q, \tag{73}$$

which, combined with Lebesgue's dominated convergence theorem, implies that

$$\|g - v_m\|^q_{L^q(K)} = \int_K |g - v_m|^q \to 0, \qquad \text{as} \quad m \to \infty. \tag{74}$$

Thus, the functions of the form (72), that is, the simple functions, are dense in $L^q(K)$. To complete the proof, it suffices to approximate simple functions by continuous functions, or simpler still, approximate the characteristic function of an arbitrary measurable set $A \subset K$ by continuous functions. By regularity of the Lebesgue measure, for any given $\varepsilon > 0$, there exist a compact set $\mathscr{K}$ and an open set $\mathscr{O} \subset \mathbb{R}^n$, such that $\mathscr{K} \subset A \subset \mathscr{O}$ and $|\mathscr{O} \setminus \mathscr{K}| < \varepsilon$, where $|\cdot|$ denotes the Lebesgue measure. Now we define

$$f(x) = \frac{\operatorname{dist}(x, \mathbb{R}^n \setminus \mathscr{O})}{\operatorname{dist}(x, \mathscr{K}) + \operatorname{dist}(x, \mathbb{R}^n \setminus \mathscr{O})}, \qquad x \in \mathbb{R}^n, \tag{75}$$

where $\operatorname{dist}(x, B) = \inf_{y \in B} |x - y|$ for any set $B \subset \mathbb{R}^n$. We have $0 \leq f \leq 1$ everywhere, $f(x) = 1$ for $x \in \mathscr{K}$ and $f(x) = 0$ for $x \in \mathbb{R}^n \setminus \mathscr{O}$. Therefore

$$\|\chi_A - f\|^q_{L^q(K)} \leq \|\chi_A - f\|^q_{L^q(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\chi_A - f|^q \leq |\mathscr{O} \setminus \mathscr{K}| < \varepsilon, \tag{76}$$

and moreover, f is continuous because of the property

$$|\operatorname{dist}(x, B) - \operatorname{dist}(y, B)| \leq |x - y|, \qquad x, y \in \mathbb{R}^n, \tag{77}$$

which holds for any set $B \subset \mathbb{R}^n$. The proof is completed. ☐

As a simple application of mollifiers, let us prove the following important result known as the fundamental lemma of calculus of variations, which is attributed to Paul du Bois-Reymond.

Lemma 22 (du Bois-Reymond)**.** *Let $u \in L^1_{\text{loc}}(\Omega)$ and let*

$$\int_\Omega u\varphi = 0 \qquad \text{for all} \quad \varphi \in \mathscr{D}(\Omega). \tag{78}$$

Then $u = 0$ almost everywhere in Ω.

Proof. Since mollification (66) is the integration against a function from $\mathscr{D}(\Omega)$ for small $\varepsilon > 0$, it follows that $u_\varepsilon(x) = 0$ eventually for each $x \in \Omega$. Let $K \subset \Omega$ be a compact set. Then u_ε converges to u in $L^1(K)$, meaning that $u = 0$ almost everywhere in K. As $K \subset \Omega$ was an arbitrary compact set, we conclude that $u = 0$ almost everywhere in Ω. ☐

In order to study differentiability properties of u_ε, we need to be able to differentiate an integral with respect to a parameter. The following result is appropriate for our purposes.

Theorem 23 (Leibniz rule, version 2)**.** *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, and let $I \subset \mathbb{R}$ be an open interval. Suppose that $f : \Omega \times I \to \mathbb{R}$ is a function satisfying*

- *$f(\cdot, t) \in L^1(\Omega)$ for each fixed $t \in I$,*
- *$f(y, \cdot) \in \mathscr{C}^1(I)$ for almost every $y \in \Omega$,*
- *There is $g \in L^1(\Omega)$ such that $|f_t(y, t)| \leq g(y)$ for almost every $y \in \Omega$ and for each $t \in I$, where f_t is the derivative of f with respect to $t \in I$.*

Then the function $F : I \to \mathbb{R}$ defined by

$$F(t) = \int_\Omega f(y, t)\mathrm{d}y \qquad (t \in I), \tag{79}$$

satisfies $F \in \mathscr{C}^1(I)$, and

$$F'(t) = \int_\Omega f_t(y, t)\mathrm{d}y, \qquad t \in I. \tag{80}$$

Proof. First, we claim that

$$G(t) = \int_\Omega f_t(y, t)\mathrm{d}y, \qquad t \in I, \tag{81}$$

is continuous in t. Since $f_t(y,t) - f_t(y,s) \to 0$ as $|t-s| \to 0$ for almost every y, it suffices to bound $|f_t(y,t) - f_t(y,s)|$ by an integrable function, uniformly in s and t. But this is exactly what we have assumed in the third bulleted item.

Now let $a \in I$ be an arbitrary but fixed point. Since G is continuous on I, from the fundamental theorem of calculus we have

$$G(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t G(s) \mathrm{d}s, \tag{82}$$

which leads to

$$\begin{aligned} \int_\Omega f_t(y,t) \mathrm{d}y &= \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \int_\Omega f_t(y,s) \mathrm{d}y \mathrm{d}s \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \int_a^t f_t(y,s) \mathrm{d}s \mathrm{d}y \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(f(y,t) - f(y,a)\right) \mathrm{d}y \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega f(y,t) \mathrm{d}y, \end{aligned} \tag{83}$$

where we have used Fubini's theorem in the second equality and the fundamental theorem of calculus for almost every $y \in \Omega$ in the third equality. □

Corollary 24. *In the context of mollification, cf.* (65) *and* (66)*, let $u \in L^1_{\text{loc}}(\Omega)$ and let $K \subset \Omega$ be a compact set. Then for all sufficiently small $\varepsilon > 0$, we have $u_\varepsilon \in \mathscr{C}^\infty(K)$ and*

$$\partial^\alpha u_\varepsilon(x) = \int_{\mathbb{R}^n} \partial^\alpha \rho_\varepsilon(x-y) u(y) \mathrm{d}y \qquad (x \in K), \tag{84}$$

for any $\alpha \in \mathbb{N}_0^n$.

Proof. What we need to show is for $\phi \in \mathscr{D}(B_\varepsilon)$ with $\varepsilon > 0$ sufficiently small,

$$\frac{\partial}{\partial x_i} \int_{\mathbb{R}^n} \phi(x-y) u(y) \mathrm{d}y = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} \phi(x-y) u(y) \mathrm{d}y \qquad (x \in K). \tag{85}$$

With $t = x_i$, the conditions are easily verified. For instance, we have

$$\left|\frac{\partial}{\partial x_i} \phi(x-y) u(y)\right| \leq |u(y)| \sup_{B_\varepsilon} |\partial_i \phi|, \tag{86}$$

which confirms the condition after the third bullet point. □

Now we can prove the main result of this section, the result known as *Weyl's lemma.*

Theorem 25 (Weyl 1940)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in H^1(\Omega)$ satisfy*

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0, \qquad \textit{for all} \quad \varphi \in \mathscr{D}(\Omega). \tag{87}$$

Then up to a modification on a set of measure zero, u is harmonic in the classical sense. In particular, we have $u \in \mathscr{C}^\omega(\Omega)$.

Proof. For $\varepsilon > 0$, let u_ε be the mollified version of u, cf. (65) and (66). Let $K \subset \Omega$ be a compact set, and let $\varepsilon > 0$ be sufficiently small. Then from Corollary 24, we know that $u_\varepsilon \in \mathscr{C}^\infty(K)$, and

$$\Delta u_\varepsilon(x) = \int \Delta \rho_\varepsilon(x-y) u(y) \mathrm{d}y \qquad (x \in K). \tag{88}$$

Developing this further, we get

$$\begin{aligned}\Delta u_\varepsilon(x) &= \int \Delta\rho_\varepsilon(x-y)u(y)\mathrm{d}y = \int \Delta_y\rho_\varepsilon(x-y)u(y)\mathrm{d}y \\ &= -\int \nabla_y\rho_\varepsilon(x-y)\cdot\nabla u(y)\mathrm{d}y = 0,\end{aligned} \tag{89}$$

where we have used integration by parts for strong derivatives in the second equality, and the property (87) in the last equality. We also have used Δ_y and ∇_y to indicate that the implied derivatives are with respect to the y variable. Hence the functions u_ε are harmonic in K.

On the other hand, Theorem 20b) tells us that $u_\varepsilon \to u$ in $L^1(K)$ as $\varepsilon \to 0$. From the mean value property, it is easy to see that $\{u_\varepsilon\}$ forms a Cauchy sequence in the uniform norm on any compact set contained in the interior of K. This shows that u_ε converges locally uniformly to some harmonic function w in the interior of K. But u_ε also converges to u in $L^1(K)$, which means that $u = w$ almost everywhere in K. As $K \subset \Omega$ was an arbitrary compact set, we conclude that $u = w$ almost everywhere in Ω, with w a harmonic function in Ω. □

Exercise 26. Show that if $u \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\int_\Omega u\Delta\varphi = 0, \qquad \text{for all} \quad \varphi \in \mathscr{D}(\Omega), \tag{90}$$

then u is harmonic in the classical sense.

5. Weak derivatives

Recall that for functions $u, v \in L^2_{\text{loc}}(\Omega)$, we say $v = \partial_i u$ strongly in L^2 if for each compact set $K \subset \Omega$, there exists a sequence $\{\phi_k\} \subset \mathscr{C}^1(K)$ such that $\phi_k \to u$ and $\partial_i\phi_k \to v$ in $L^2(K)$. Strong derivatives are defined in terms of *approximation*. In order to show that a particular function is strongly differentiable by using the definition directly, one needs to construct a suitable approximating sequence, cf. Example 9. On the other hand, in the process of showing that strong derivatives are unique, in Lemma 10 we proved that strong derivatives satisfy an integration by parts formula, namely

$$\int_\Omega \varphi\partial_i u = -\int_\Omega u\partial_i\varphi, \qquad \varphi \in \mathscr{C}^1_c(\Omega). \tag{91}$$

We can turn this around and introduce a new concept of derivative, which is *a priori* more general than strong derivatives.

Definition 27. For $u, v \in L^1_{\text{loc}}(\Omega)$, we say $v = \partial_i u$ in the *weak sense*, or that v is a *weak derivative* of u, if

$$\int_\Omega v\varphi = -\int_\Omega u\partial_i\varphi, \tag{92}$$

for all $\varphi \in \mathscr{D}(\Omega)$.

Weak derivatives are defined in terms of *duality*. It is immediate from the du Bois-Reymond lemma that the weak derivatives are unique.

Example 28. a) Let us try to find the weak derivative of $u(x) = |x|$, $x \in \mathbb{R}$. We have

$$\begin{aligned}\int_{\mathbb{R}} |x|\varphi'(x)\mathrm{d}x &= -\int_{-\infty}^0 x\varphi'(x)\mathrm{d}x + \int_0^\infty x\varphi'(x)\mathrm{d}x \\ &= \int_{-\infty}^0 \varphi(x)\mathrm{d}x - \int_0^\infty \varphi(x)\mathrm{d}x \\ &= -\int_{\mathbb{R}} \varphi(x)\,\mathrm{sign}(x)\mathrm{d}x,\end{aligned} \tag{93}$$

for $\varphi \in \mathscr{D}(\mathbb{R})$, implying that $|x|' = \operatorname{sign}(x)$ in the weak sense. Note that as expected, the result is the same as that of Example 9.

b) Suppose that $v \in L^1_{\text{loc}}(\mathbb{R})$ is the weak derivative of sign. Then we would have

$$\int_{\mathbb{R}} v(x)\varphi(x)\mathrm{d}x = -\int_{\mathbb{R}} \varphi'(x)\operatorname{sign}(x)\mathrm{d}x = -\int_0^\infty \varphi'(x)\mathrm{d}x + \int_{-\infty}^0 \varphi'(x)\mathrm{d}x = 2\varphi(0), \tag{94}$$

for $\varphi \in \mathscr{D}(\mathbb{R})$. In particular, it is true for $\varphi \in \mathscr{D}(\mathbb{R} \setminus \{0\})$, which by the du Bois-Reymond lemma implies that $v = 0$ almost everywhere in $\mathbb{R} \setminus \{0\}$. This of course means that $v = 0$ almost everywhere in $\mathbb{R}$, and for such functions, the integral in the left hand side of (94) is equal to 0. Hence (94) cannot be satisfied if $\varphi(0) \neq 0$, meaning that the sign function is *not* weakly differentiable.[2]

The following theorem shows that in the L^2-context, strong and weak derivatives coincide.

Theorem 29 (Friedrichs 1944)**.** *Let $u, v \in L^2_{\text{loc}}(\Omega)$. Then $v = \partial_i u$ in the strong L^2-sense if and only if $v = \partial_i u$ in the weak sense.*

Proof. The integration by parts formula (91) that we proved in Lemma 10 shows that if $v = \partial_i u$ in the strong L^2-sense, then $v = \partial_i u$ also in the weak sense.

Let $v = \partial_i u$ in the weak sense, and let $K \subset \Omega$ be a compact set. We will employ the technique of mollifiers, cf. (65) and (66). Let u_ε and v_ε be the mollified versions of u and v, respectively. We know that $u_\varepsilon \to u$ and $v_\varepsilon \to v$ in $L^2(K)$ as $\varepsilon \to 0$. What remains is to show that $\partial_i u_\varepsilon \to v$ in $L^2(K)$ as $\varepsilon \to 0$, but it follows from

$$\begin{aligned} \partial_i u_\varepsilon(x) &= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}\rho_\varepsilon(x-y)u(y)\mathrm{d}y = -\int_{\mathbb{R}^n} \frac{\partial}{\partial y_i}\rho_\varepsilon(x-y)u(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)v(y)\mathrm{d}y = v_\varepsilon(x), \end{aligned} \tag{95}$$

where in the third equality we have used the fact that v is the weak derivative of u. □

Definition 30. We define the *Sobolev space* $W^{1,2}(\Omega)$ as

$$W^{1,2}(\Omega) = \{u \in L^2(\Omega) : \partial_i u \in L^2(\Omega),\, i = 1, \ldots, n\}, \tag{96}$$

and equip it with the norm $\|\cdot\|_{H^1}$.

Theorem 31 (Meyers-Serrin 1964)**.** *For $\Omega \subset \mathbb{R}^n$ open, $\mathscr{C}^\infty(\Omega) \cap H^1(\Omega)$ is dense in $W^{1,2}(\Omega)$. In particular, we have $H^1(\Omega) = W^{1,2}(\Omega)$.*

Proof. Let $u \in W^{1,2}(\Omega)$, and let $\varepsilon > 0$. We will show that there exists $\phi \in \mathscr{C}^\infty(\Omega)$ such that $\|u - \phi\|_{H^1} \leq \varepsilon$. Consider a sequence $\{\Omega_k\}$ of bounded domains, such that $\Omega = \bigcup_k \Omega_k$ and $\overline{\Omega}_k \subset \Omega_{k+1}$ for $k = 1, 2, \ldots$. Moreover, for each k, let χ_k be a smooth nonnegative function satisfying $\operatorname{supp}\chi_k \subset \Omega_{k+2} \setminus \Omega_k$, and globally, $\sum_k \chi_k \equiv 1$ in Ω. Then for each k, we define $\phi_k = (\chi_k u)_{\varepsilon_k}$ by mollification, with $\varepsilon_k > 0$ so small that $\operatorname{supp}\phi_k \subset \Omega_{k+3} \setminus \Omega_{k-1}$ (with the convention $\Omega_0 = \varnothing$) and $\|\phi_k - \chi_k u\|_{H^1} \leq \varepsilon/2^k$. This is possible because $\chi_k u \in W^{1,2}(\Omega)$ and $\partial_i \phi_k = (\partial_i(\chi_k u))_{\varepsilon_k}$. Finally, we define $\phi = \sum_k \phi_k$. There is no issue of convergence because the sum is locally finite. We have

$$\|u - \phi\|_{H^1} \leq \|\textstyle\sum_k(\chi_k u - \phi_k)\|_{H^1} \leq \sum_k \|\chi_k u - \phi_k\|_{H^1} \leq \varepsilon, \tag{97}$$

which establishes the proof. □

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[2]However, we have $\operatorname{sign}' = 2\delta$ in the sense of distributions. From this perspective, the reason why sign is not weakly differentiable is that *by definition* weak derivatives are locally integrable functions and δ is not a locally integrable function.

6. Boundary values of weak solutions

To summarize what we have accomplished so far on the Dirichlet problem with the Sobolev space approach, for any given $g \in H^1(\Omega)$ with $\Omega \subset \mathbb{R}^n$ a bounded domain, we have constructed a harmonic function $u \in H^1(\Omega)$ satisfying $u - g \in H^1_0(\Omega)$. We know from Weyl's lemma that u is harmonic in the classical sense in Ω.

The condition $u - g \in H^1_0(\Omega)$ is supposed to be a generalized form of the Dirichlet boundary condition $(u-g)|_{\partial\Omega} = 0$. We want to clarify what it would mean, at least when $\partial\Omega$ is not so irregular. To get some insight, let us consider the one dimensional case first.

Lemma 32. *Let $u \in H^1_0(\Sigma)$, with $\Sigma = (0,1)$. Then there is $w \in \mathscr{C}(\bar{\Sigma})$ with $w(0) = w(1) = 0$ such that $u = w$ almost everywhere in Σ.*

Proof. There exists a sequence $\{u_k\} \subset \mathscr{D}(\Sigma)$ such that $u_k \to u$ in H^1. From the fundamental theorem of calculus, for $v \in \mathscr{D}(\Sigma)$ and for $0 < h < 1$ we have

$$v(h) = \int_0^h v'(t)\mathrm{d}t, \tag{98}$$

which implies that

$$|v(h)|^2 \leq h \int_0^h |v'(t)|^2 \mathrm{d}t \leq h\|v'\|^2_{L^2}. \tag{99}$$

Applying this inequality to the differences $u_j - u_k$, we conclude that $\{u_k\}$ is Cauchy in the uniform norm on I and that $u_k \to w$ uniformly for some $w \in \mathscr{C}(I)$. This means that $u = w$ almost everywhere. We want to see if the boundary value $w(0)$ can be defined. By continuity, we have

$$|w(h)| \leq \sqrt{h}\|u'\|_{L^2} \leq \sqrt{h}\|u\|_{H^1}, \tag{100}$$

and so

$$w(0) = \lim_{h\to 0} w(h) = 0, \tag{101}$$

establishing the lemma. □

Now we look at the two dimensional case, where a new phenomenon arises.

Lemma 33. *Let $\Sigma = (0,1)$ and $Q = \Sigma \times \Sigma$. For $0 < h < 1$, define $\gamma_h : \mathscr{D}(Q) \to \mathscr{D}(\Sigma)$ by $(\gamma_h\varphi)(x) = \varphi(x,h)$. Then γ_h can be uniquely extended to a bounded map $\gamma_h : H^1_0(Q) \to L^2(\Sigma)$, and for $u \in H^1_0(Q)$, we have $\gamma_h u \to 0$ in $L^2(\Sigma)$ as $h \to 0$.*

Proof. For $v \in \mathscr{D}(Q)$ and for $0 < h < 1$ we have

$$v(x,h) = \int_0^h \partial_y v(x,t)\mathrm{d}t, \tag{102}$$

which implies that

$$|v(x,h)|^2 \leq h \int_0^h |\partial_y v(t)|^2 \mathrm{d}t \leq h \int_0^1 |\partial_y v(x,t)|^2 \mathrm{d}t, \tag{103}$$

and upon integrating over x, that

$$\int_0^1 |v(x,h)|^2 \mathrm{d}x \leq h \int_Q |\partial_y v(x,t)|^2 \mathrm{d}t\mathrm{d}x \leq h\|\nabla v\|^2_{L^2(Q)}. \tag{104}$$

This means that $\|\gamma_h v\|_{L^2(\Sigma)} \leq \sqrt{h}\|v\|_{H^1(Q)}$ and that γ_h can be uniquely extended to a bounded map $\gamma_h : H^1_0(Q) \to L^2(\Sigma)$. □

The map γ_h in the preceding lemma is called the *trace map*, in the sense that functions defined on Q leave their traces on the lower dimensional manifold $\Sigma\times\{h\}$. Then the *boundary trace* $\gamma_0 u$ of u (onto $\Sigma\times\{0\}$) is defined in terms of the limit $\gamma_h u$ as $h\to 0$. If we piece together the boundary traces of u onto the four edges of Q, we get the boundary trace of u onto the whole boundary ∂Q, as an element of $L^2(\partial Q)$. Thus $u-g\in H^1_0(Q)$ implies that the trace of $u-g$ onto ∂Q vanishes in the L^2 sense, or equivalently, that the traces of u and g onto ∂Q are equal to each other as elements of $L^2(\partial Q)$.

Example 34. Let $u\in\mathscr{D}(Q)$ be a function with $\gamma_h u\not\equiv 0$ for some $0<h<1$, and let $\phi\in\mathscr{D}(\mathbb{R})$ be a function satisfying $\phi(0)=1$ and $0\leq\phi\leq 1$. Then $v_k(x,y)=u(x,y)[1-\phi(k(y-h))]$ satisfies $v_k\in\mathscr{D}(Q)$ and $\gamma_h v_k=0$. Moreover, it is easy to see that $v_k\to u$ in $L^2(Q)$, because the area of the region on which v_k differs from u shrinks to 0. This shows that the trace map γ_h cannot be extended to $L^2(Q)$ as a continuous map, because $\gamma_h v_k=0$ for all k, while $\gamma_h u_k=\gamma_h u\not\equiv 0$ for the constant sequence $u_k=u$.

Example 35. Let $\phi\in\mathscr{D}(\mathbb{R})$ be an even function with $\phi(0)=1$, and let $u(\rho,\varphi,z)=\phi(\rho/z)$ be defined in the region $\{0<z<1\}\subset\mathbb{R}^3$ in cylindrical coordinates. Then $\gamma_h u=u|_{z=h}$ satisfies $\|\gamma_h u\|_{L^2(\mathbb{R}^2)}\to 0$ as $h\to 0$, because

$$\|\gamma_h u\|^2_{L^2(\mathbb{R}^2)}=2\pi\int_0^\infty|\phi(\rho/h)|^2\rho\mathrm{d}\rho=2\pi h^2\int_0^\infty|\phi(t)|^2\,t\mathrm{d}t. \tag{105}$$

However, $(\gamma_h u)(0)=1$ for all $h>0$, hence $\gamma_h u$ does not go to 0 pointwise. This is an example where the boundary trace vanishes in the L^2-sense, but does not vanish pointwise. Moreover, we have $u\in H^1(\{0<z<1\})$, since

$$\int_0^\infty|\partial_\rho u|^2\rho\mathrm{d}\rho=z^{-2}\int_0^\infty|\phi'(\rho/z)|^2\rho\mathrm{d}\rho=\int_0^\infty|\phi'(t)|^2\,t\mathrm{d}t, \tag{106}$$

and

$$\int_0^\infty|\partial_z u|^2\rho\mathrm{d}\rho=z^{-4}\int_0^\infty|\phi'(\rho/z)|^2\rho^3\mathrm{d}\rho=\int_0^\infty|\phi'(t)|^2\,t^3\mathrm{d}t. \tag{107}$$

Exercise 36. Find a function $u\in H^1_0(\mathbb{H})$ where $\mathbb{H}\subset\mathbb{R}^2$ is the upper half plane, whose boundary trace vanishes in the L^2-sense, but does not vanish pointwise.

The general case is not more complicated than the two dimensional case.

Theorem 37. *Let $Q=(0,1)^n$ and $\Sigma_h=(0,1)^{n-1}\times\{h\}$. Let $\Omega\subset\mathbb{R}^n$ be a domain, and let $\Phi:\bar{Q}\to\bar{\Omega}$ be an injective $\mathscr{C}^1$ map, satisfying $\Phi(Q)\subset\Omega$ and $\Phi(\Sigma_0)\subset\partial\Omega$. With $\Gamma_h=\Phi(\Sigma_h)$ for $0<h<1$, define the trace map $\gamma_h:\mathscr{D}(\Omega)\to C(\Gamma_h)$ by $\gamma_h\varphi=\varphi|_{\Gamma_h}$. Then γ_h can be uniquely extended to a bounded map $\gamma_h:H^1_0(\Omega)\to L^2(\Gamma_h)$, and moreover, for $u\in H^1_0(\Omega)$ we have $\|\gamma_h u\|_{L^2(\Gamma_h)}\to 0$ as $h\to 0$.*

Proof. Let $X=\{v\in\mathscr{C}^1(\bar{Q}):v|_{\Sigma_0}=0\}$, and define $\hat{\gamma}_h:X\to C(\Sigma_h)$ by $\hat{\gamma}_h v=v|_{\Sigma_h}$. Then as in the preceding lemma, we have $\|\hat{\gamma}_h v\|_{L^2(\Sigma_h)}\leq\sqrt{h}\|\nabla v\|_{L^2(Q)}$ for $v\in X$. Now let $u\in\mathscr{D}(\Omega)$. Then the pull-back $\hat{u}=\Phi^*u$ defined by $\hat{u}(\hat{x})=u(\Phi(\hat{x}))$ satisfies $\hat{u}\in X$. Moreover, from the transformation properties of the first derivatives, we have

$$\|\nabla\hat{u}\|_{L^2(Q)}\leq c\|\nabla u\|_{L^2(\Omega)},\qquad\text{where}\quad c=\sup_Q|\det D\Phi|^{-\frac12}|D\Phi|, \tag{108}$$

and $|D\Phi|$ is the spectral norm of the Jacobian matrix $D\Phi$. We also have

$$\|\gamma_h u\|_{L^2(\Gamma_h)}\leq c'\|\hat{\gamma}_h\hat{u}\|_{L^2(\Sigma_h)}, \tag{109}$$

where c' depends only on the Jacobian $D\Phi$. Combining all three estimates, we infer

$$\|\gamma_h u\|_{L^2(\Gamma_h)}\leq C\sqrt{h}\|\nabla u\|_{L^2(\Omega)}, \tag{110}$$

and the theorem follows. □

Finally, we include a complementary result which basically says that if a function $u \in H^1_0(\Omega)$ is continuous at a boundary point $z \in \partial\Omega$, then $u(z) = 0$. In the next chapter, we will see that the weak solutions to $\Delta u = 0$ are continuous up to the boundary, under some regularity conditions on the boundary $\partial\Omega$.

Lemma 38 (Nirenberg 1955)**.** *In the setting of the preceding theorem, let $u \in H^1_0(\Omega)$, and let u be continuous at $z \in \Phi(\Sigma_0)$. Then $u(z) = 0$.*

Proof. Without loss of generality we assume that $\Omega = \{x \in \mathbb{R}^n : |x| < 1,\, x_n > 0\}$ and $z = 0$. Let $u \in \mathscr{D}(\Omega)$, and with $h > 0$, let $E = B \times (0, h) \subset \Omega$ be a cylinder, where $B \subset \mathbb{R}^{n-1}$ is a ball centred at 0 whose volume is $|B| = h$. For $x \in E$, we have

$$|u(x)| = \Big| \int_0^{x_n} \partial_n u(x', t)\mathrm{d}t \Big| \leq \int_0^h |\partial_n u(x', t)|\mathrm{d}t, \tag{111}$$

where $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Now using the Cauchy-Bunyakowsky-Schwarz inequality and squaring, we get

$$|u(x)|^2 \leq h \int_0^h |\partial_n u(x', t)|^2 \mathrm{d}t, \tag{112}$$

which, upon integrating along $x' = \text{const}$, gives

$$\int_0^h |u(x', t)|^2 \mathrm{d}t \leq h^2 \int_0^h |\partial_n u(x', t)|^2 \mathrm{d}t. \tag{113}$$

Then we integrate over $x' \in B$, and obtain

$$\int_E |u|^2 \leq h^2 \int_E |\partial_n u|^2 \leq h^2 \int_E |\nabla u|^2 = |E| \int_E |\nabla u|^2, \tag{114}$$

which means

$$\frac{1}{|E|} \int_E |u|^2 \leq \int_E |\nabla u|^2. \tag{115}$$

The same inequality is true for $u \in H^1_0(\Omega)$ by density, and the right hand side goes to 0 as $h \to 0$ by the fact that $u \in H^1(\Omega)$. Since u is continuous at 0, the left hand side goes to $|u(0)|^2$ as $h \to 0$, which proves the lemma. □

7. Problems and exercises

1. In the context of Example 3, construct a minimizing sequence of piecewise smooth functions satisfying the homogeneous boundary condition, which diverges in a set that is dense in $\mathbb{D}$. Show that this sequence converges to $u \equiv 0$ in L^2.

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u \in \mathscr{C}^2(\Omega)$ satisfy

$$E_*(u) := \int_\Omega (|\nabla u|^2 + |u|^2) < \infty.$$

Prove the following.

(a) If $\Delta u = u$ in Ω, then $E_*(u + v) > E_*(u)$ for all nontrivial $v \in \mathscr{C}^1_c(\Omega)$.
(b) Conversely, if $E_*(u + v) \geq E_*(u)$ for all $v \in \mathscr{C}^1_c(\Omega)$, then $\Delta u = u$ in Ω.

3. Let $v(x) = \log \log(2/|x|)$ and let $v_k \in \mathscr{C}(\mathbb{D})$ be defined by $v_k(x) = \min\{k, v(x)\}$. Show that the norms $\|v_k\|_{H^1(\mathbb{D})}$ are uniformly bounded. Exhibit a sequence $\{u_k\} \subset \tilde{\mathscr{C}}^1(\mathbb{D})$ that is Cauchy with respect to the norm $\|\cdot\|_{H^1(\mathbb{D})}$, whose limit is not essentially bounded.

4. Let $u_1 \in H^1(\Omega)$ and $u_2 \in H^1(\Omega)$ be the weak solutions of $\Delta u = u$ satisfying, respectively, $u_1 - g_1 \in H^1_0(\Omega)$ and $u_2 - g_2 \in H^1_0(\Omega)$, where $g_1, g_2 \in H^1(\Omega)$. Show that

$$\|u_1 - u_2\|_{H^1} \leq \|g_1 - g_2\|_{H^1}. \tag{116}$$

5. Let Ω be a bounded domain, and let $g \in H^1(\Omega)$. Show that there is a unique $u \in H^1(\Omega)$ satisfying $u - g \in H^1_0(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla v = 0 \qquad \text{for all} \quad v \in H^1_0(\Omega). \tag{117}$$

6. Prove that the function u given by the Poisson formula for the Dirichlet problem on a ball, say, B_r, is harmonic in B_r for boundary data $g \in L^1(\partial B_r)$, and takes correct boundary values wherever g is continuous.

7. Let u be given by the Poisson formula for the Dirichlet problem on the unit ball $B = B_1$, for boundary data $g \in L^p(\partial B)$, with $1 \leq p < \infty$. Show that u satisfies the boundary condition $u|_{\partial B} = g$ in the L^p-sense, i.e., that $u_r \to g$ in $L^p(\partial B)$ as $r \to 1$, where $u_r(x) = u(rx)$ for $x \in \partial B$ and $0 \leq r < 1$.

8. Show that Theorem 23 is true when Ω is an arbitrary complete measure space. Moreover, in the context of the theorem, replace the conditions on f by

- $f(\cdot, t) \in L^1(\Omega)$ for almost every $t \in I$,
- $f(y, \cdot)$ is absolutely continuous on I, for almost every $y \in \Omega$,
- $f_t \in L^1(\Omega \times I)$,

and prove that F is absolutely continuous on I and $F' = G$ almost everywhere on I, where G is as in (81).

9. Show that if $u \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\int_\Omega u \Delta \varphi = 0, \qquad \text{for all} \quad \varphi \in \mathscr{D}(\Omega), \tag{118}$$

then u is harmonic in the classical sense.

10. Find a function $u \in H^1_0(\mathbb{H})$ where $\mathbb{H} \subset \mathbb{R}^2$ is the upper half plane, whose boundary trace vanishes in the L^2-sense, but does not vanish pointwise.

11. Let $\Omega \subset \mathbb{R}^n$ be an open set, let $k \geq 0$ be an integer, and let $1 \leq p \leq \infty$. Then the *Sobolev space* $W^{k,p}(\Omega)$ by definition consists of those $u \in L^p(\Omega)$ such that $\partial^\alpha u \in L^p(\Omega)$ for each α with $|\alpha| \leq k$. Equip it with the norm

$$\|u\|_{W^{k,p}(\Omega)} = N(\{\|\partial^\alpha u\|_{L^p(\Omega)} : |\alpha| \leq k\}),$$

where N is a norm on the finite dimensional space $\{\lambda_\alpha \in \mathbb{R} : |\alpha| \leq k\}$.

a) Show that the topology of $W^{k,p}(\Omega)$ does not depend on the choice of N.
b) Show that $W^{k,p}(\Omega)$ is a Banach space for any $k \geq 0$ and $1 \leq p \leq \infty$.
c) Prove that $\mathscr{D}(\mathbb{R}^n)$ is a dense subspace of $W^{k,p}(\mathbb{R}^n)$, for any $k \geq 0$ and $1 \leq p < \infty$.

12. Let $Q = (0,1)^n$ and let $Q_h = (h, 1-h)^n$. For $h > 0$ small, define the trace map $\gamma_h : C^1(Q) \to C(\partial Q_h)$ by $\gamma_h v = v|_{\partial Q_h}$.

a) Prove that γ_h can be uniquely extended to a bounded map $\gamma_h : H^1(Q) \to L^2(\partial Q_h)$.
b) Make sense of the boundary trace $\gamma_0 u = \lim\limits_{h \to 0} \gamma_h u$ in $L^2(\partial Q)$ for $u \in H^1(Q)$.
c) Show that $\gamma_0 u = 0$ for $u \in H^1_0(Q)$.
d) Let $u \in H^1_0(Q)$ and let u be continuous at 0. Show that $u(0) = 0$.

13. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $W^{1,1}_{\text{loc}}(\Omega)$ be the set of locally integrable functions whose (weak) derivatives are locally integrable (that is, in $L^1_{\text{loc}}(\Omega)$).

a) Show that if $u, v \in W^{1,1}_{\text{loc}}(\Omega)$ and $uv, u\partial_i v + v\partial_i u \in L^1_{\text{loc}}(\Omega)$, then $uv \in W^{1,1}_{\text{loc}}(\Omega)$ and $\partial_i(uv) = u\partial_i v + v\partial_i u$.
b) Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism between Ω and Ω'. Show that if $u \in W^{1,1}_{\text{loc}}(\Omega')$ then $v = u \circ \phi \in W^{1,1}_{\text{loc}}(\Omega)$ and $\partial_i v(x) = \sum_j \partial_i \phi_j(x)(\partial_j u)(\phi(x))$, where ϕ_j is the j-th component of ϕ, and $(\partial_j u)(\phi(x))$ is the evaluation of $\partial_j u$ at the point $\phi(x)$.
c) Let $f \in C^1(\mathbb{R})$ with both f and f' bounded, and let $u \in W^{1,1}_{\text{loc}}(\Omega)$. Prove that $f \circ u \in W^{1,1}_{\text{loc}}(\Omega)$ and that $\partial_i(f \circ u) = (f' \circ u)\partial_i u$.
d) Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ and let $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$ pointwise. Prove that $\partial_i u^+ = \theta(u)\partial_i u$ and $\partial_i u^- = \theta(-u)\partial_i u$ *a.e.*, where θ is the Heaviside step function. In particular, show that $|u| \in W^{1,p}(\Omega)$ if $u \in W^{1,p}(\Omega)$.