MATH 580 ASSIGNMENT 5

DUE FRIDAY NOVEMBER 30

1. Show that the embedding $H^1(B) \hookrightarrow L^q(B)$ is *not* compact, where $B \subset \mathbb{R}^n$ is an open ball, and $q = \frac{2n}{n-2}$.
2. Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain, and let $1 \leq p < \infty$. Let $S \subset W^{1,p}(\Omega)$ be a set bounded in $W^{1,p}(\Omega)$, and suppose that $K_1 \subset K_2 \subset \ldots \subset \Omega$ is a sequence of compact sets satisfying $\Omega = \bigcup_j K_j$ and

$$\|u\|_{L^p(\Omega\setminus K_j)} \to 0 \qquad \text{as} \quad j \to \infty,$$

uniformly in $u \in S$. Prove that S is relatively compact in $L^p(\Omega)$. Is there an unbounded domain $\Omega \subset \mathbb{R}^n$ for which the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact? If yes, provide an explicit example. If no, give a proof.
3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the *Poincaré inequality*

$$\int_\Omega |u - \bar{u}|^2 \leq C \int_\Omega |\nabla u|^2, \qquad \text{where} \qquad \bar{u} = \int_\Omega u, \tag{$*$}$$

that is hypothesized to hold for all $u \in H^1(\Omega)$, and for some constant $C = C(\Omega) > 0$.
   (a) Is there a bounded domain Ω for which C is infinite?
   (b) Characterize the best constant $C > 0$ appearing in the Poincaré inequality via an eigenvalue problem.
   (c) Find the best constant when Ω is the rectangle $\Omega = (0, a) \times (0, b)$. Exhibit a function u that achieves the equality in $(*)$.
4. Completely solve the problem $-\Delta u = \lambda u$ with the homogeneous Dirichlet boundary condition, in the disk $D_r = \{x^2 + y^2 < r^2\} \subset \mathbb{R}^2$. You are encouraged to consult extra material outside what is covered in the lectures.
5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let

$$a(u, v) = \int_\Omega (a_{ij}\partial_i u \partial_j v + buv),$$

where the repeated indices are summer over, and the coefficients a_{ij} and b are smooth functions in $\bar{\Omega}$, with a symmetric matrix $[a_{ij}]$ satisfying the uniform ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \qquad \xi \in \mathbb{R}^n, \quad x \in \bar{\Omega},$$

for some constant $\alpha > 0$. We define the map $\tilde{A} : H^1_0(\Omega) \to [H^1_0(\Omega)]'$ by $\langle \tilde{A}u, v\rangle = a(u, v)$ for $u, v \in H^1_0(\Omega)$, and then we let A be the unbounded operator in $L^2(\Omega)$ that is given

Date: Fall 2018.

by the restriction of $\tilde{A}$ onto $L^2(\Omega)$, i.e., $Au = \tilde{A}u$ for $u \in \mathrm{Dom}(A)$ where

$$\mathrm{Dom}(A) = \{u \in H_0^1(\Omega) : \tilde{A}u \in L^2(\Omega)\}.$$

Consider the generalized eigenvalue problem

$$Au = \lambda \rho u,$$

where ρ is a (fixed) positive smooth function in $\bar{\Omega}$. Prove the following, by using the spectral theorem for compact self-adjoint positive operators where possible.

(a) The eigenvalues $\{\lambda_k\}$ are countable and real, and that $\lambda_k \to \infty$ as $k \to \infty$. Each eigenvalue has a finite multiplicity.

(b) The eigenfunctions $\{u_k\}$ form a complete orthonormal system in $L^2(\Omega)$, with respect to a suitable inner product.

(c) The system $\{u_k\}$ is complete and orthogonal in $H_0^1(\Omega)$, with respect to the inner product $a(u, v) + t \int_\Omega \rho uv$, where t is a suitably chosen constant.

(d) The eigenfunctions are smooth in Ω, and are smooth up to the boundary if $\partial\Omega$ is smooth.