MATH 580 ASSIGNMENT 4

DUE FRIDAY NOVEMBER 16

1. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $W^{1,1}_{\text{loc}}(\Omega)$ be the set of locally integrable functions whose (weak) derivatives are locally integrable (that is, in $L^1_{\text{loc}}(\Omega)$).
   a) Show that if $u, v \in W^{1,1}_{\text{loc}}(\Omega)$ and $uv, u\partial_i v + v\partial_i u \in L^1_{\text{loc}}(\Omega)$, then $uv \in W^{1,1}_{\text{loc}}(\Omega)$ and $\partial_i(uv) = u\partial_i v + v\partial_i u$.
   b) Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism between Ω and Ω'. Show that if $u \in W^{1,1}_{\text{loc}}(\Omega')$ then $v = u \circ \phi \in W^{1,1}_{\text{loc}}(\Omega)$ and $\partial_i v(x) = \sum_j \partial_i \phi_j(x)(\partial_j u)(\phi(x))$, where ϕ_j is the j-th component of ϕ, and $(\partial_j u)(\phi(x))$ is the evaluation of $\partial_j u$ at the point $\phi(x)$.
   c) Let $f \in C^1(\mathbb{R})$ with both f and f' bounded, and let $u \in W^{1,1}_{\text{loc}}(\Omega)$. Prove that $f \circ u \in W^{1,1}_{\text{loc}}(\Omega)$ and that $\partial_i(f \circ u) = (f' \circ u)\partial_i u$.
   d) Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ and let $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$ pointwise. Prove that $\partial_i u^+ = \theta(u)\partial_i u$ and $\partial_i u^- = \theta(-u)\partial_i u$ *a.e.*, where θ is the Heaviside step function. In particular, show that $|u| \in W^{1,p}(\Omega)$ if $u \in W^{1,p}(\Omega)$.
2. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and consider the bilinear form

$$a(u, v) = \int_\Omega (a_{ij}\partial_i u \partial_j v + buv),$$

   where the repeated indices are summer over, and the coefficients a_{ij} and b are smooth functions on $\bar{\Omega}$, with a_{ij} satisfying the uniform ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \xi \in \mathbb{R}^n, \quad x \in \bar{\Omega},$$

   for some constant $\lambda > 0$.
   a) Show that the mapping $A : H^1(\Omega) \to [H^1(\Omega)]'$, defined by $\langle Au, v\rangle = a(u, v)$, is bounded, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $[H^1(\Omega)]'$ and $H^1(\Omega)$.
   b) Show that if $b > 0$ in $\bar{\Omega}$, then

$$\langle Au, u\rangle \geq \alpha \|u\|^2_{H^1}, \qquad u \in H^1(\Omega),$$

   for some constant $\alpha > 0$.
   c) Supposing that the condition in b) holds, show that given $f \in L^2(\Omega)$, there exists a unique function $u \in H^1(\Omega)$ satisfying $a(u, v) = \int_\Omega fv$ for all $v \in H^1(\Omega)$.
   d) Suppose that $u \in H^1(\Omega)$ is sufficiently smooth and satisfies $a(u, v) = \int_\Omega fv$ for all $v \in H^1(\Omega)$. What differential equation does u satisfy in Ω? What boundary condition does u satisfy?

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3. In the setting of the preceding question, let a_{ij} be constant functions, and suppose that $u \in H_0^1(\Omega)$ satisfies

$$a(u,v) = \int_\Omega fv \qquad \text{for all } v \in \mathscr{D}(\Omega),$$

   a) Show that if $f \in H^k_{\text{loc}}(\Omega)$ then $u \in H^{k+2}_{\text{loc}}(\Omega)$.
   b) How large k should be in order to guarantee that the classical equation holds?
   c) Prove that if $f \in H^k(\Omega)$ then $u \in H^{k+2}(\Omega)$.
   d) What condition would guarantee that the boundary condition is satisfied classically?
   e) Show that if f and b are analytic in Ω, then u is analytic in Ω.
4. Recall that the *Sobolev inequality*

$$\|u\|_{L^q} \leq C\|u\|_{W^{1,p}}, \qquad u \in W^{1,p}(\mathbb{R}^n), \tag{$*$}$$

   with some constant $C = C(p,q)$, is valid when $1 \leq p \leq q < \infty$, and $\frac{1}{p} \leq \frac{1}{q} + \frac{1}{n}$.
   a) By way of a counterexample, show that the inequality $(*)$ fails whenever $q < p$.
   b) Show that $(*)$ fails when $\frac{1}{p} > \frac{1}{q} + \frac{1}{n}$.
   c) Show that $(*)$ fails for $p = n$ and $q = \infty$ when $n \geq 2$.
   d) Using $(*)$ as a basis, derive sufficient conditions on the exponents p, q, k, m under which the inequality

$$\|u\|_{W^{m,q}} \leq C\|u\|_{W^{k,p}}, \qquad u \in W^{k,p}(\mathbb{R}^n),$$

   is valid.