MATH 580 ASSIGNMENT 3

DUE WEDNESDAY OCTOBER 31

1. In this exercise we will study Sobolev spaces on the interval $I = (0, 1)$. Let $1 \leq p < \infty$, and define the norm
$$\|u\|_{1,p} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{1/p},$$
for $u \in C^1(\bar{I})$. Let $H^{1,p}(I)$ be the completion of $C^1(\bar{I})$ with respect to the norm $\|\cdot\|_{1,p}$.
a) Show that there is a continuous injection of $H^{1,p}(I)$ into $L^p(I)$.
b) Prove the *Sobolev inequality*
$$\|u\|_{L^\infty} \leq 2^{1-1/p}\|u\|_{1,p}, \qquad u \in H^{1,p}(I).$$
c) Since $H^{1,p}(I)$ is a subspace of $L^p(I)$, an element of $H^{1,p}(I)$ is an equivalence class of functions that differ on sets of measure zero. So one can change the values of a function on a set of measure zero, and it would still correspond to the same element in $H^{1,p}(I)$. Make sense of, and prove the statement that the elements of $H^{1,p}(I)$ are continuous functions.
d) Prove the *Friedrichs inequality*
$$\|u\|_{L^p}^p \leq 2^{p-1}\|u'\|_{L^p}^p + 2^{p-1}|u(\xi)|^p, \qquad u \in H^{1,p}(I), \quad \xi \in [0, 1].$$
In particular, make sense of the derivative u' appearing in the right hand side.
e) Prove the *Poincaré inequality*
$$\|u\|_{L^p}^p \leq 2^{p-1}\|u'\|_{L^p}^p + 2^{p-1}\Big|\int_I u\Big|^p, \qquad u \in H^{1,p}(I).$$
f) Let $H_0^{1,p}(I)$ be the closure of $C_c^1(I)$ in $H^{1,p}(I)$. Show that
$$H_0^{1,p}(I) = \{u \in H^{1,p}(I) : u(0) = u(1) = 0\}.$$
g) With u' understood in the weak sense, let
$$W^{1,p}(I) = \{u \in L^p(I) : u' \in L^p(I)\}.$$
Prove (a special case of) the *Meyers-Serrin theorem*: $H^{1,p}(I) = W^{1,p}(I)$.
2. a) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Prove that there is a constant $c > 0$ with the following property: Given any $g \in H^1(\Omega)$ and any constant $t > -c$, there exists a unique $u \in H^1(\Omega)$ satisfying $u - g \in H_0^1(\Omega)$ and
$$\int_\Omega \big(\nabla u \cdot \nabla v + tuv\big) = 0 \qquad \text{for all} \quad v \in \mathscr{D}(\Omega). \tag{$*$}$$
What classical equation does $(*)$ correspond to?

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b) In the context of a), for a fixed $t > -c$, let us denote the map that sends g to u by

$$S : g \mapsto u : H^1(\Omega) \to H^1(\Omega).$$

Show that S is linear and is Lipschitz continuous in the sense that there is a (real) constant M such that

$$\|S(g_1) - S(g_2)\|_{H^1} \leq M\|g_1 - g_2\|_{H^1}, \qquad g_1, g_2 \in H^1(\Omega).$$

3. Find a function $u \in H^1_0(U) \cap C(U)$ where $U \subset \mathbb{R}^2$ is the upper half plane, whose boundary trace vanishes in the L^2-sense, but does not vanish pointwise.
4. Let u be given by the Poisson formula for the Dirichlet problem on the unit ball $B = B_1$, for a boundary datum $g \in L^p(\partial B)$, with $1 \leq p < \infty$.
   a) Show that u is harmonic in B, and takes correct boundary values (in the classical sense) wherever g is continuous.
   b) Show that u satisfies the boundary condition $u|_{\partial B} = g$ in the L^p-sense, i.e., that $u_r \to g$ in $L^p(\partial B)$ as $r \to 1$, where $u_r(x) = u(rx)$ for $x \in \partial B$ and $0 \leq r < 1$.
5. Let $Q = (0,1)^n$ and let $Q_h = (h, 1-h)^n$. For $h > 0$ small, define the trace map $\gamma_h : C^1(Q) \to C(\partial Q_h)$ by $\gamma_h v = v|_{\partial Q_h}$.
   a) Prove that γ_h can be uniquely extended to a bounded map $\gamma_h : H^1(Q) \to L^2(\partial Q_h)$.
   b) Make sense of the boundary trace $\gamma_0 u = \lim_{h\to 0} \gamma_h u$ in $L^2(\partial Q)$ for $u \in H^1(Q)$.
   c) Show that $\gamma_0 u = 0$ for $u \in H^1_0(Q)$.
   d) Let $u \in H^1_0(Q)$ and let u be continuous at 0. Show that $u(0) = 0$.