MATH 580 ASSIGNMENT 2

DUE FRIDAY OCTOBER 12

1. Let Ω be a bounded domain in $\mathbb{R}^n$.
   (a) Show that if the Dirichlet problem in Ω is solvable for any boundary condition $g \in C(\partial\Omega)$, then each boundary point $z \in \partial\Omega$ admits a barrier.
   (b) Is regularity of a boundary point a local property? In other words, if $z \in \partial\Omega$ is regular, and if Ω' is a domain that coincides with Ω in a neighbourhood of z (hence in particular $z \in \partial\Omega'$), then is z also regular as a point on $\partial\Omega'$?
2. (Poincaré 1887) In this exercise, we will implement Poincaré's *method of sweeping out* (*méthode de balayage*) to solve the Dirichlet problem. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $g \in C(\Omega)$. Suppose that $u_0 \in C(\bar{\Omega})$ is a function subharmonic in Ω and $u_0 = g$ on $\partial\Omega$. The idea is to iteratively improve the initial approximation u_0 towards a harmonic function by solving the Dirichlet problem on a suitable sequence of balls.
   (a) Show that there exist countably many open balls B_k such that $\Omega = \bigcup_k B_k$.
   (b) Consider the sequence $B_1, B_2, B_1, B_2, B_3, B_1, \ldots$, so that each B_k is occurring infinitely many times, and let us reuse the notation B_k to denote the k-th member of this sequence. Then we define the functions $u_1, u_2, \ldots \in C(\bar{\Omega})$ by the following recursive procedure: For $k = 1, 2, \ldots$, put $u_k = u_{k-1}$ in $\Omega \setminus B_k$, and let u_k be the solution of $\Delta u_k = 0$ in B_k with the boundary condition $u_{k-1}|_{\partial B_k}$. Prove that $u_k \to u$ locally uniformly in Ω, for some $u \in C^\infty(\Omega)$ that is harmonic in Ω.
   (c) Show that if there exists $v \in C(\bar{\Omega})$ satisfying $\Delta v = 0$ in Ω and $v = g$ on $\partial\Omega$, then indeed $u = v$, where u is the function we constructed in (b). So if there exists a solution, then our method would produce the same solution. However, we want to demonstrate existence without any prior assumption on existence.
   (d) Prove that if there exists a barrier at $z \in \partial\Omega$, then $u(x) \to g(z)$ as $\Omega \ni x \to z$, where u is the function we constructed in (b). Recall that a function $\varphi \in C(\bar{\Omega})$ is called a *barrier for Ω at $z \in \partial\Omega$* if
      - φ is subharmonic in Ω,
      - $\varphi(z) = 0$,
      - $\varphi < 0$ in $\bar{\Omega} \setminus \{z\}$.

   We call the boundary point $z \in \partial\Omega$ *regular* if there is a barrier for Ω at $z \in \partial\Omega$.
   (e) Assuming that all boundary points are regular, this procedure reduces the Dirichlet problem into the problem of constructing a subharmonic function u_0 with $u_0|_{\partial\Omega} = g$. Instead of constructing such u_0 for the given g directly, let us approximate g by functions for which such a construction is simpler. Show that if $\{v_j\} \subset C(\bar{\Omega})$ is

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a sequence with $\Delta v_j = 0$ in Ω and $v_j \to g$ uniformly on $\partial\Omega$, then there exists a function $u \in C(\bar{\Omega})$ satisfying $\Delta u = 0$ in Ω and $u = g$ on $\partial\Omega$.

(f) Show that any polynomial can be written as the difference of two subharmonic functions in Ω. Hence it suffices to extend g into a continuos function on $\bar{\Omega}$, and approximate the resulting function by polynomials (explain why). State what standard results we need in order to realize this.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u \in C^2(\Omega)$ satisfy

$$E_*(u) := \int_\Omega (|\nabla u|^2 + |u|^2) < \infty.$$

Prove the followings.

(a) If $\Delta u = u$ in Ω, then $E_*(u+v) > E_*(u)$ for all nontrivial $v \in C^1_c(\Omega)$.

(b) Conversely, if $E_*(u+v) \geq E_*(u)$ for all $v \in C^1_c(\Omega)$, then $\Delta u = u$ in Ω.

4. Consider the problem of minimizing the energy

$$Q(u) = \int_I \left(1 + |u'(x)|^2\right)^{\frac{1}{4}} \,\mathrm{d}x,$$

for all $u \in C^1(I) \cap C(\bar{I})$ satisfying $u(0) = 0$ and $u(1) = 1$, where $I = (0, 1)$. Show that the infimum of Q over the admissible functions is 1, but this value is not assumed by any admissible function.

5. Consider the Dirichlet problem on the unit disk $\mathbb{D} \subset \mathbb{R}^2$ with the homogeneous Dirichlet boundary condition. The solution is $u \equiv 0$, which also minimizes the Dirichlet energy

$$E(u) = \int_{\mathbb{D}} |\nabla u|^2.$$

Construct a sequence $\{u_k\} \subset C(\bar{\mathbb{D}})$ of functions satisfying all of the following conditions.

- u_k is piecewise smooth and $u_k|_{\partial\mathbb{D}} = 0$ for all k,
- $E(u_k) \to 0$ as $k \to \infty$, and
- u_k diverges as $k \to \infty$ in a set that is dense in $\mathbb{D}$.

Then show that $u_k \to 0$ in $H^1(\mathbb{D})$.

6. Exhibit a sequence $\{v_k\} \subset \tilde{C}^1(\mathbb{D})$ that is Cauchy with respect to the H^1-norm, whose limit is not essentially bounded.