MATH 580 ASSIGNMENT 1

DUE FRIDAY SEPTEMBER 28

1. Let $f : B_R \setminus \{0\} \to \mathbb{R}$ and let there be a constant $M > 0$ such that
$$\int_{B_R \setminus B_\varepsilon} |f| \leq M \qquad \text{for any } \varepsilon > 0, \tag{1}$$
where the integral is understood in the Riemann sense. Then the *improper Riemann integral* of f over B_R is defined to be
$$\int_{B_R} f = \lim_{\varepsilon \to 0} \int_{B_R \setminus B_\varepsilon} f, \tag{2}$$
and we say that f is *absolutely integrable.*
   (a) Let $\{U_\varepsilon\}$ be a family of open sets in $\mathbb{R}^n$, satisfying $U_\varepsilon \subset U_\delta$ for $\varepsilon < \delta$ and $\bigcap_\varepsilon U_\varepsilon = \{0\}$. Show that if f is absolutely integrable, then
$$\lim_{\varepsilon \to 0} \int_{B_R \setminus U_\varepsilon} f = \lim_{\varepsilon \to 0} \int_{B_R \setminus B_\varepsilon} f, \tag{3}$$
meaning that the improper Riemann integral does not depend on the family $\{U_\varepsilon\}$.
   (b) Show that if f is absolutely integrable, then the Lebesgue integral of f over B_R exists, and is equal to the improper Riemann integral (2).
2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the boundary value problem
$$\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 1 \quad \text{on } \partial\Omega. \tag{4}$$
Prove the following.
   a) Any solution of (4) in $C^2(\bar{\Omega})$ with $f(u) = u^m$ where $m \in \mathbb{N}$ is odd, must satisfy $0 \leq u \leq 1$ in $\bar{\Omega}$, and is unique.
   b) The only solution of (4) in $C^2(\bar{\Omega})$ with $f(u) = u - u^{-1}$ is $u \equiv 1$.
3. Let $g \in \mathscr{C}(\partial\mathbb{D})$, and let $u : \mathbb{D} \to \mathbb{R}$ be given by
$$u(y) = -\frac{1}{\pi} \int_{\partial\mathbb{D}} g(x) \log|x - y| \, \mathrm{d}x, \tag{5}$$
Show that u is harmonic in $\mathbb{D}$, and continuous in $\bar{\mathbb{D}}$, with $\int_{\partial\mathbb{D}} u = 0$. Moreover, prove that $\partial_r u(y) \to g(x)$ as $\mathbb{D} \ni y \to x \in \partial\mathbb{D}$, where ∂_r is the radial derivative.
4. Let $u \in \mathscr{C}^2(\mathbb{R}^n)$ be an entire harmonic function in $\mathbb{R}^n$. Prove the following.
   (a) Any tangent hyperplane to the graph of u intersects the graph more than once.
   (b) If u satisfies $u(x) \geq -C(1 + |x|)^m$ for some constants C and $m \in \mathbb{N}$, then u is a polynomial of degree less or equal to m.

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5. Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $u \in C^1(\Omega)$ and that for each $y \in \Omega$ there exists $r^* > 0$ such that
$$\int_{\partial B_r} \partial_\nu u = 0,$$
for all $0 < r < r^*$, where ∂_ν is the normal derivative. Show that u is harmonic in Ω.
6. Let $\Omega \subset \mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n : x_n > 0\}$ be a domain, and let $\Sigma = \{x \in \partial\Omega : x_n = 0\}$ be a nonempty open subset of the hyperplane $\partial\mathbb{R}^n_+ \equiv \{x_n = 0\}$. Prove the following.
   (a) Suppose that $u \in C^2(\Omega) \cap C(\Omega \cup \Sigma)$ is harmonic in Ω and $u = 0$ on Σ. Denote by x^* the reflection $(x_1, \dots, x_{n-1}, -x_n)$ of $x = (x_1, \dots, x_{n-1}, x_n)$, and let
   $$\tilde{\Omega} = \Omega \cup \Sigma \cup \{x^* : x \in \Omega\}.$$
   Then the function $\tilde{u} \in C(\tilde{\Omega})$ defined by $\tilde{u} = u$ in $\Omega \cup \Sigma$ and $\tilde{u}(x^*) = -u(x)$ for $x \in \Omega$ is harmonic in $\tilde{\Omega}$. This result is known as the *Schwarz reflection principle.*
   (b) The Cauchy problem for the Laplace equation $\Delta u = 0$ with the Cauchy data $u = 0$ and $\partial_n u = g$ on the hyperplane $\{x_n = 0\}$ has no solution in any neighbourhood of $0 \in \mathbb{R}^n$, if g is not analytic at $0 \in \mathbb{R}^{n-1}$.
   (c) A bounded harmonic function in $\mathbb{R}^n_+$ with $u = 0$ on $\{x_n = 0\}$ is identically zero.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.