MATH 580 FALL 2014 PRACTICE PROBLEMS

NOVEMBER 30, 2014

1. Prove the removable singularity theorem for harmonic functions in two dimensions.
2. Let p be a nontrivial polynomial of n variables, and let f be a real analytic function in a neighbourhood of $0 \in \mathbb{R}^n$. Then show that there is a neighbourhood of $0 \in \mathbb{R}^n$, on which the equation $p(\partial)u = f$ has a solution. Supposing that $p(\xi) = \sum_\alpha a_\alpha \xi^\alpha$, here the operator $p(\partial)$ is given by
$$p(\partial) = \textstyle\sum_\alpha a_\alpha \partial^\alpha.$$
3. Here we will prove a version of the maximum principle for unbounded domains and for functions that are not necessarily continuous up to the boundary.
   (a) Exhibit a harmonic function violating the (weak) maximum principle on an unbounded domain.
   (b) Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $K_1 \subset K_2 \subset \ldots$ be a nested sequence of compact subsets of Ω, such that $\bigcup_j K_j = \Omega$. Suppose that $u \in C(\Omega)$ is a subharmonic function satisfying
   $$\limsup_{j\to\infty} \sup_{x\in\partial K_j} u(x) \leq 0.$$
   Show that $u \leq 0$ in Ω.
   (c) Let $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. We topologize it by identifying it with the n-sphere through a stereographical projection (which is required to be a homeomorphism by definition). In particular, we have $x_j \to \infty$ if $\{x_j\}$ eventually escapes any compact set of $\mathbb{R}^n$. For $\Omega \subset \mathbb{R}^n$ a domain, we denote its boundary in $\hat{\mathbb{R}}^n$ by $\hat{\partial}\Omega$. Show that $\hat{\partial}\Omega = \partial\Omega$ if Ω is bounded, and $\hat{\partial}\Omega = \partial\Omega \cup \{\infty\}$ if Ω is unbounded. Suppose that $u \in C(\Omega)$ is a subharmonic function satisfying
   $$\limsup_{\Omega\ni x\to z} u(x) \leq 0, \qquad \text{for each} \quad z \in \hat{\partial}\Omega.$$
   Show that $u \leq 0$ in Ω.
4. In this exercise, we will study a general version of Perron's method. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $g : \hat{\partial}\Omega \to \mathbb{R}$ be a function. Then we define the *Perron family*
$$S_g = \{v \in C(\Omega) : v \text{ subharmonic in } \Omega, \limsup_{\Omega\ni x\to z} u(x) \leq g(z) \text{ for each } z \in \hat{\partial}\Omega\},$$
and the *Perron solution*
$$u(x) = \sup_{v\in S_g} v(x), \qquad x \in \Omega.$$
We call a subharmonic function $\varphi \in C(\Omega)$ a *barrier at* $z \in \hat{\partial}\Omega$ if $\varphi(x) \to 0$ as $\Omega \ni x \to z$, and $\sup_{\Omega\setminus B_\delta(z)} \varphi < 0$ for each $\delta > 0$.
   (a) Show that if g is bounded, then u is well-defined.
   (b) Prove that if u is well-defined, then $\Delta u = 0$ in Ω.
   (c) Supposing that u is well-defined, prove that if there is a barrier at $z \in \hat{\partial}\Omega$ and if g is continuous at z, then $u(x) \to g(z)$ as $\Omega \ni x \to z$.
   (d) Show that for unbounded domains in dimensions more than two, the point ∞ is always regular.