PERRON'S METHOD FOR THE DIRICHLET PROBLEM

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Abstract. We present here the classical method of subharmonic functions for solving the Dirichlet problem that culminated in the works of Perron and Wiener.

Contents

1. Brief history of the Dirichlet problem

Given a domain $\Omega \subset \mathbb{R}^n$ and a function $g : \partial\Omega \to \mathbb{R}$, the *Dirichlet problem* is to find a function u satisfying

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In the previous set of notes, we established that uniqueness holds if Ω is bounded and g is continuous. We have also seen that the Dirichlet problem has a solution if Ω is a ball.

The Dirichlet problem turned out to be fundamental in many areas of mathematics and physics, and the efforts to solve this problem led directly to many revolutionary ideas in mathematics. The importance of this problem cannot be overstated.

The first serious study of the Dirichlet problem on general domains with general boundary conditions was done by George Green in his *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, published in 1828. He reduced the problem into a problem of constructing what we now call Green's functions, and argued that Green's function exists for any domain. His methods were not rigorous by today's standards, but the ideas were highly influential in the subsequent developments. It should be noted that George Green had basically no formal schooling when he wrote the Essay, and most probably he never knew the real importance of his discovery, as the Essay went unnoticed by the community until 1845, four years after Green's death, when William Thomson rediscovered it.

The next idea came from Gauss in 1840. He noticed that given a function ρ on $\partial\Omega$, the so-called single layer potential

$$u(y) = (V\rho)(y) \equiv \int_{\partial\Omega} E_y \rho, \tag{2}$$

is harmonic in Ω, and hence that if we find ρ satisfying $V\rho = g$ on $\partial\Omega$, the Dirichlet problem would be solved. Informally, we want to arrange electric charges on the surface $\partial\Omega$ so that the resulting electric potential is equal to g on $\partial\Omega$. If we imagine that $\partial\Omega$ is made of a good conductor, then in the absence of an external field, the equilibrium configuration of charges

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on the surface will be the one that produces constant potential throughout $\partial\Omega$. The same configuration also minimizes the electrostatic energy

$$E(\rho) = \frac{1}{2}\int_{\partial\Omega} \rho V\rho, \tag{3}$$

among all ρ such that the net charge $\int_{\partial\Omega}\rho$ is fixed. In order to solve $V\rho = g$, we imagine that there is some external electric field whose potential at the surface coincides with $-g$. The equilibrium configuration in this case would satisfy $V\rho - g = \text{const}$, and minimize the energy

$$E(\rho) = \frac{1}{2}\int_{\partial\Omega} \rho V\rho - \int_{\partial\Omega} g\rho, \tag{4}$$

among all ρ such that the net charge is fixed. Then we would have $V(\rho-\rho') = g$ for some ρ' satisfying $V\rho' = \text{const}$, or more directly, we can simply add a suitable constant to $u = V\rho$ to solve the Dirichlet problem. Gauss did not prove the existence of a minimizer to (4), but he remarked that it was obvious.

Around 1847, that is just after Green's work became widely known, William Thomson (Lord Kelvin) and Gustav Lejeune-Dirichlet suggested to minimize the energy

$$E(u) = \int_{\Omega} |\nabla u|^2, \tag{5}$$

subject to $u|_{\partial\Omega} = g$. Note that by Green's first identity we have

$$E(u) = \int_{\partial\Omega} u\partial_\nu u - \int_{\Omega} u\Delta u, \tag{6}$$

which explains why $E(u)$ can be considered as the energy of the configuration, since in view of Green's formula

$$u(y) = \int_{\Omega} E_y \Delta u + \int_{\partial\Omega} u\partial_\nu E_y - \int_{\partial\Omega} E_y \partial_\nu u, \tag{7}$$

$\partial_\nu u$ is the surface charge density, and $-\Delta u$ is the volume charge density, that produce the field u. This and other considerations seemed to show that the Dirichlet problem is equivalent to minimizing the energy $E(u)$ subject to $u|_{\partial\Omega} = g$. Moreover, since $E(u) \geq 0$ for any u, the existence of u minimizing $E(u)$ appeared to be obvious. Riemann called these two statements the *Dirichlet principle*, and used it to prove his fundamental mapping theorem, in 1851. However, starting around 1860, the Dirichlet principle in particular and calculus of variations at the time in general went under serious scrutiny, most notably by Karl Weierstrass and Riemann's former student Friedrich Prym. Weierstrass argued that even if E is bounded from below, it is possible that the infimum is never attained by an admissible function, in which case there would be no admissible function that minimizes the energy. He backed his reasoning by an explicit example of an energy that has no minimizer. Let us look at his example.

Example 1 (Weierstrass 1870)**.** Consider the problem of minimizing the energy

$$Q(u) = \int_I x^2 |u'(x)|^2 \,\mathrm{d}x, \tag{8}$$

for all $u \in C(\overline{I})$ with piecewise continuous derivatives in I, satisfying the boundary conditions $u(-1) = 0$ and $u(1) = 1$, where $I = (-1, 1)$. The infimum of E over the admissible functions is 0, because obviously $E \geq 0$ and for the function

$$v(x) = \begin{cases} 0 & \text{for } x < 0, \\ x/\delta & \text{for } 0 < x < \delta, \\ 1 & \text{for } x > \delta, \end{cases} \tag{9}$$

we have $E(v) = \frac{\delta}{3}$, which can be made arbitrarily small by choosing $\delta > 0$ small. However, there is no admissible function u for which $E(u) = 0$, since this would mean that $u(x) = 0$ for $x < 0$ and $u(x) = 1$ for $x > 0$.

In 1871, Prym constructed a striking example of a continuous function g on the boundary of a disk, such that there is not a single function u with finite energy that equals g on the boundary. This makes it impossible even to talk about a minimizer since all functions with the correct boundary condition would have infinite energy. The following is a a variation of Prym's example, and is constructed by Jacques Hadamard.

Example 2 (Hadamard 1906)**.** Let $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$, and let $u : \mathbb{D} \to \mathbb{R}$ be given in polar coordinates by

$$u(r,\theta) = \sum_{n=1}^{\infty} n^{-2} r^{n!} \sin(n!\theta). \tag{10}$$

It is easy to check that each term of the series is harmonic, and the series converges absolutely uniformly in $\overline{\mathbb{D}}$. Hence u is harmonic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. On the other hand, we have

$$E(u) = \int_{\mathbb{D}} |\nabla u|^2 \geq \int_0^{2\pi}\int_0^{\rho} |\partial_r u(r,\theta)|^2 r \,\mathrm{d}r\,\mathrm{d}\theta = \sum_{n=1}^{\infty} \frac{\pi n!}{2n^4}\rho^{2n!} \geq \sum_{n=1}^{m} \frac{\pi n!}{2n^4}\rho^{2n!}, \tag{11}$$

for any $\rho < 1$ and any integer m. This implies that $E(u) = \infty$. To conclude, there exists a Dirichlet datum $g \in C(\partial\mathbb{D})$ for which the Dirichlet problem is perfectly solvable, but the solution cannot be obtained by minimizing the Dirichlet energy. There is no full equivalence between the Dirichlet problem and the minimization problem.

Now that the Dirichlet principle is not reliable anymore, it became an urgent matter to solve the Dirichlet problem to "rescue" the Riemann mapping theorem. By 1870, Weierstrass' former student Hermann Schwarz had largely succeeded in achieving this goal. He solved the Dirichlet problem on polygonal domains by an explicit formula, and used an iterative approximation process to extend his results to an arbitrary planar region with piecewise analytic boundary. His approximation method is now known as the *Schwarz alternating method*, and is one of the popular methods to solve boundary value problems on a computer.

The next advance was Carl Neumann's work of 1877, that was based on the earlier work of August Beer from 1860. The idea was similar to Gauss', but instead of the single layer potential, Beer suggested the use of the double layer potential

$$u(y) = (K\mu)(y) \equiv \int_{\partial\Omega} \mu\partial_\nu E_y. \tag{12}$$

The function u is automatically harmonic in Ω, and the requirement $u|_{\partial\Omega} = g$ is equivalent to the integral equation $(1-2K)\mu = 2g$ on the boundary. This equation was solved by Neumann in terms of the series

$$(1-2K)^{-1} = 1 + 2K + (2K)^2 + \ldots, \tag{13}$$

which bears his name now. Neumann showed that the series converges if Ω is a 3 dimensional convex domain whose boundary does not consist of two conical surfaces. The efforts to solve the equation $(1-2K)\mu = 2g$ in cases the above series does not converge, led Ivar Fredholm to his discovery of Fredholm theory in 1900.

Since the analyticity or convexity conditions on the boundary seemed to be rather artificial, the search was still on to find a good method to solve the general Dirichlet problem. Then in 1887, Henri Poincaré published a paper introducing a very flexible method with far reaching consequences. Poincaré started with a subharmonic function that has the correct boundary values, and repeatedly solved the Dirichlet problem on small balls to make the function more and more nearly harmonic. He showed that the process converges if the succession of balls is chosen carefully, and produces a harmonic function in the interior. Moreover, this harmonic function assumes correct boundary values, if each point on the boundary of the domain can be touched from outside by a nontrivial sphere. The process is now called Poincaré's *sweeping out process* or the *balayage method*.

Poincaré's work made the Dirichlet problem very approachable, and invited further work on weakening the conditions on the boundary. For instance, it led to the work of William Fogg Osgood, published in 1900, in which the author establishes solvability of the Dirichlet problem in very general planar domains. While the situation was quite satisfactory, there had essentially been no development as to the validity of the original Dirichlet principle, until 1899, when David Hilbert gave a rigorous justification of the Dirichlet principle under some assumptions on the boundary of the domain. This marked the beginning of a major program to put calculus of variations on a rigorous foundation.

During that period it was generally believed that the assumptions on the boundary of the domain that seemed to be present in all available results were due to limitations of the methods employed, rather than being inherent in the problem. It was Stanisław Zaremba who first pointed out in 1911 that there exist regions in which the Dirichlet problem is not solvable, even when the boundary condition is completely reasonable.

Example 3 (Zaremba 1911)**.** Let $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disk, and consider the domain $\Omega = \mathbb{D} \setminus \{0\}$. The boundary of Ω consists of the circle $\partial\mathbb{D}$ and the point $\{0\}$. Consider the Dirichlet problem $\Delta u = 0$ in Ω, with the boundary conditions $u \equiv 0$ on $\partial\mathbb{D}$ and $u(0) = 1$. Suppose that there exists a solution. Then u is harmonic in Ω, and continuous in $\mathbb{D}$ with $u(0) = 1$. Since u is bounded in Ω, one can extend u continuously to $\mathbb{D}$ so that the resulting function is harmonic in $\mathbb{D}$. By uniqueness for the Dirichlet problem in $\mathbb{D}$, the extension must identically be equal to 0, because $u \equiv 0$ on $\partial\mathbb{D}$. However, this contradicts the fact that u is continuous in $\mathbb{D}$ with $u(0) = 1$. Hence there is no solution to the original problem. In other words, the boundary condition at $x = 0$ is simply "ignored".

One could argue that Zaremba's example is not terribly surprising because the boundary point 0 is an isolated point. However, in 1913, Henri Lebesgue produced an example of a 3 dimensional domain whose boundary consists of a single connected piece. This example will be studied in §3, Example 11. The time period under discussion is now 1920's, which saw intense developments in the study of the Dirichlet problem, then known as potential theory, powered by the newly founded Lebesgue integration theory and functional analytic point of view. Three basic approaches were most popular: Poincaré-type methods which use subharmonic functions, integral equation methods based on potential representations of harmonic functions, and finally, variational methods related to minimizing the Dirichlet energy. While the former two would still be considered as part of potential theory, the third approach has since separated because of its distinct Hilbert space/variational flavour. In what follows, we will study the method of subharmonic functions in detail.

2. Perron's method

In this section, we will discuss the method discovered by Oskar Perron in 1923, as a simpler replacement of the Poincaré process. Recall that we want to solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{14}$$

In what follows, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, and that $g : \partial\Omega \to \mathbb{R}$ is a bounded function. Recall that a continuous function $u \in C(\Omega)$ is called *subharmonic* in Ω, if for any $y \in \Omega$, there exists $r^* > 0$ such that

$$u(y) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u, \qquad 0 < r < r^*. \tag{15}$$

Let us denote by $\mathfrak{Sub}(\Omega)$ the set of subharmonic functions on Ω. The following properties will be useful.

- If $u \in \mathfrak{Sub}(\Omega)$ and if $u(z) = \sup_\Omega u$ for some $z \in \Omega$, then u is constant.
- If $u \in \mathfrak{Sub}(\Omega) \cap C(\overline{\Omega})$, $v \in C(\overline{\Omega})$ is harmonic in Ω, and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.
- If $u_1, u_2 \in \mathfrak{Sub}(\Omega)$ then $\max\{u_1, u_2\} \in \mathfrak{Sub}(\Omega)$.
- If $u \in \mathfrak{Sub}(\Omega)$ and if $\bar{u} \in C(\Omega)$ satisfies $\Delta\bar{u} = 0$ in B and $\bar{u} = u$ in $\Omega \setminus B$ for some $B \subset \Omega$, then $\bar{u} \in \mathfrak{Sub}(\Omega)$.

The first two properties are simply the strong and weak maximum principles. The third property is clear from

$$u_i(y) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u_i \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} \max\{u_1, u_2\}, \qquad i = 1, 2. \tag{16}$$

For the last property, we only need to check (15) for $y \in \partial B$, as

$$\bar{u}(y) = u(y) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} \bar{u}. \tag{17}$$

To proceed further, we define the *Perron (lower) family*

$$S_g = \{v \in \mathfrak{Sub}(\Omega) \cap C(\overline{\Omega}) : v|_{\partial\Omega} \leq g\}, \tag{18}$$

and the *Perron (lower) solution*

$$u(x) = (P_\Omega g)(x) = \sup_{v \in S_g} v(x), \qquad x \in \Omega. \tag{19}$$

Any constant function c satisfying $c \leq g$ is in S_g, so $S_g \neq \varnothing$. Moreover, any $v \in S_g$ satisfies $v \leq \sup_{\partial\Omega} g$, hence the Perron solution u is well-defined. We will show that the Perron solution is a solution of the Dirichlet problem, under some mild regularity assumptions on the boundary of Ω. Before doing so, let us perform a consistency check. Suppose that $\Delta w = 0$ in Ω and $w|_{\partial\Omega} = g$. Then obviously $w \in S_g$. Also, the weak maximum principle shows that any $v \in S_g$ satisfies $v \leq w$ pointwise. Therefore we must have $u = w$.

Theorem 4 (Perron 1923)**.** *For the Perron solution $u = P_\Omega g$, we have $\Delta u = 0$ in Ω.*

Proof. Let $B_r(x)$ be a nonempty open ball whose closure is in Ω, and let $\{u_k\} \subset S_g$ be a sequence satisfying $u_k(x) \to u(x)$ as $k \to \infty$. Without loss of generality, we can assume that the sequence is nondecreasing, e.g., by replacing u_k by $\max\{u_1, \ldots, u_k\}$. For each k, let $\bar{u}_k \in C(\Omega)$ be the function harmonic in $B_r(x)$ which agrees with u_k in $\Omega \setminus B_r(x)$. We have $u_k \leq \bar{u}_k$, and $\bar{u}_k \in S_g$ hence $\bar{u}(x) \leq u(x)$, so $\bar{u}_k(x) \to u(x)$ as well. The sequence $\{\bar{u}_k\}$ is also nondecreasing, so by Harnack's second convergence theorem, there exists a harmonic function $\bar{u}$ in $B_r(x)$ such that $\bar{u}_k \to \bar{u}$ locally uniformly in $B_r(x)$. In particular, we have $\bar{u}(x) = u(x)$.

We want to show that $u = \bar{u}$ in $B_r(x)$, which would then imply that u is harmonic in Ω. Pick $y \in B_r(x)$, and let $\{\tilde{u}_k\} \subset S_g$ be a sequence satisfying $\tilde{u}_k(y) \to u(y)$. Without loss of generality, we can assume that the sequence is nondecreasing, that $\bar{u}_k \leq \tilde{u}_k$, and that $\tilde{u}_k$ is harmonic in $B_r(x)$. Again by Harnack's theorem, there exists a harmonic function $\tilde{u}$ in $B_r(x)$ such that $\tilde{u}_k \to \tilde{u}$ locally uniformly in $B_r(x)$, and we have $\tilde{u}(y) = u(y)$. Because of the arrangement $\bar{u}_k \leq \tilde{u}_k$, we get $\bar{u} \leq \tilde{u}$ in $B_r(x)$, and in addition taking into account that $\tilde{u}_k \leq u$ and that $\bar{u}_k(x) \to u(x)$, we infer $\tilde{u}(x) = u(x)$. So $\bar{u} - \tilde{u}$ is harmonic and nonpositive in $B_r(x)$, while $\bar{u}(x) - \tilde{u}(x) = 0$. Then the strong maximum principle gives $\bar{u} = \tilde{u}$ in $B_r(x)$, which implies that $\bar{u}(y) = u(y)$. As $y \in B_r(x)$ was arbitrary, $u = \bar{u}$ in $B_r(x)$. □

Now we need to check if u satisfies the required boundary condition $u|_{\partial\Omega} = g$. Let $z \in \partial\Omega$, and let us try to imagine what can go wrong so that $u(x) \not\to g(z)$ as $x \to z$. It is possible that $\liminf_{x\to z} u(x) < g(z)$, or $\limsup_{x\to z} u(x) > g(z)$, or both. To rule out the first scenario, it suffices to show that there is a sequence $\{w_k\} \in S_g$ such that $w_k(z) \to g(z)$. Indeed, since $u \geq w_k$ pointwise, we would have $\liminf_{x\to z} u(x) \geq w_k(z)$ for each k. The existence of such a sequence $\{w_k\}$ means, in a certain sense, that the domain Ω is able to support a sufficiently rich family of subharmonic functions. In a similar fashion, to rule out the second scenario, we need to have a sufficiently rich family of superharmonic functions, and as superharmonic functions are simply the negatives of subharmonic functions, it turns out that both scenarios can be handled by the same method. We start by introducing the concept of a barrier.

Definition 5. A function $\varphi \in C(\overline{\Omega})$ is called a *barrier for* Ω *at* $z \in \partial\Omega$ if

- $\varphi \in \mathfrak{Sub}(\Omega)$,
- $\varphi(z) = 0$,
- $\varphi < 0$ on $\partial\Omega \setminus \{z\}$.

We call the boundary point $z \in \partial\Omega$ *regular* if there is a barrier for Ω at $z \in \partial\Omega$.

Lemma 6. *Let $z \in \partial\Omega$ be a regular point, and let g be continuous at z. Then for any given $\varepsilon > 0$, there exists $w \in S_g$ such that $w(z) \geq g(z) - \varepsilon$.*

Proof. Let $\varepsilon > 0$, and let φ be a barrier at z. Then there exists $\delta > 0$ such that $|g(x)-g(z)| < \varepsilon$ for $x \in \partial\Omega \cap B_\delta(z)$. Choose $M > 0$ so large that $M\varphi(x) + 2\|g\|_\infty < 0$ for $x \in \partial\Omega \setminus B_\delta(z)$, and consider the function $w = M\varphi + g(z) - \varepsilon$. Obviously, $w \in \mathfrak{Sub}(\Omega) \cap C(\overline{\Omega})$ and $w(z) = g(z) - \varepsilon$. Moreover, we have

$$M\varphi(x) + g(z) - \varepsilon < M\varphi(x) + g(x) \leq g(x), \qquad x \in \partial\Omega \cap B_\delta(z), \tag{20}$$

and

$$M\varphi(x) + g(z) - \varepsilon < -2\|g\|_\infty + g(z) \leq g(x), \qquad x \in \partial\Omega \setminus B_\delta(z), \tag{21}$$

which imply that $w \in S_g$. □

Exercise 1. Why is regularity of a boundary point a local property? In other words, if $z \in \partial\Omega$ is regular, and if Ω' is a domain that coincides with Ω in a neighbourhood of z (hence in particular $z \in \partial\Omega'$), then can you conclude that z is also regular as a point of $\partial\Omega'$?

Exercise 2. Show that if the Dirichlet problem in Ω is solvable for any boundary condition $g \in C(\partial\Omega)$, then each $z \in \partial\Omega$ is a regular point.

The following theorem implies the converse to the preceding exercise: If all boundary points are regular, then the Dirichlet problem is solvable for any $g \in C(\partial\Omega)$.

Theorem 7 (Perron 1923)**.** *Assume that $z \in \partial\Omega$ is a regular point, and that g is continuous at z. Then we have $u(x) \to g(z)$ as $\Omega \ni x \to z$.*

Proof. By Lemma 6, for any $\varepsilon > 0$ there exists $w \in S_g$ such that $w(z) \geq g(z) - \varepsilon$. By definition, we have $u \geq w$ in Ω. This shows that

$$\liminf_{\Omega\ni x\to z} u(x) \geq g(z) - \varepsilon, \tag{22}$$

and as $\varepsilon > 0$ was arbitrary, the same relation is true with $\varepsilon = 0$. On the other hand, again by Lemma 6, for any $\varepsilon > 0$ there exists $w \in S_{-g}$ such that $w(z) \geq -g(z) - \varepsilon$. Let $v \in S_g$. Then $v + w \in \mathfrak{Sub}(\Omega) \cap C(\overline{\Omega})$ and $v + w \leq 0$ on $\partial\Omega$. This means that $v \leq -w$ in Ω. Since v is an arbitrary element of S_g, the same inequality is true for u, hence

$$\limsup_{\Omega\ni x\to z} u(x) \leq g(z) + \varepsilon, \tag{23}$$

and as $\varepsilon > 0$ was arbitrary, the same relation is true with $\varepsilon = 0$. □

Exercise 3. Let us modify the definition of a barrier (Definition 5) by allowing $\varphi \in C(\Omega)$ and replacing the condition after the third bullet point therein by

$$\limsup_{\Omega\ni x\to y} \varphi(x) < 0 \qquad \text{for each} \quad y \in \partial\Omega \setminus \{z\}. \tag{24}$$

Show that Theorem 7 is still valid when the regularity concept is accordingly modified. $\oslash$

Corollary 8. *Green's function exists for the domain Ω if each point of $\partial\Omega$ is regular.*

3. Boundary regularity

It is of importance to derive simple criteria for a boundary point to admit a barrier. The following is referred to as *Poincaré's criterion* or the *exterior sphere condition.*

Theorem 9 (Poincaré 1887)**.** *Suppose that $B_r(y)\cap\Omega = \varnothing$ and $\overline{B_r(y)}\cap\partial\Omega = \{z\}$, with $r > 0$. Then z is a regular point.*

Proof. For $n \geq 3$, we claim that

$$\varphi(x) = \frac{1}{|x-y|^{n-2}} - \frac{1}{r^{n-2}}, \qquad x \in \overline{\Omega}, \tag{25}$$

is a barrier at z. Indeed, φ is harmonic in $\mathbb{R}^n \setminus \{y\}$, $\varphi(z) = 0$, and $\varphi(x) < 0$ for $x \in \mathbb{R}^n \setminus \overline{B_r(y)}$. For $n = 2$, it is again straightforward to check that

$$\varphi(x) = \log\frac{1}{|x-y|} - \log\frac{1}{r}, \qquad x \in \overline{\Omega}, \tag{26}$$

is a barrier at z. $\square$

Remark 10. In fact, we have the following criterion due to Lebesgue: The point $0 \in \partial\Omega$ is regular if any $x \in \Omega$ near 0 satisfies $x_n < f(|x'|)$, where $x' = (x_1,\ldots,x_{n-1})$ and $f(r) = ar^{1/m}$ for some $a > 0$ and $m > 0$. The case $m = 1$ is known as *Zaremba's criterion* or the *exterior cone condition.*

The following example shows that Lebesgue's criterion is nearly optimal in the sense that the criterion would not be valid if $f(r) = a/\log\frac{1}{r}$.

Example 11 (Lebesgue 1913)**.** Let $I = \{(0,0,s) : 0 \leq s \leq 1\} \subset \mathbb{R}^3$ and let

$$v(x) = \int_0^1 \frac{s\,\mathrm{d}s}{|x-p(s)|} \qquad x \in \mathbb{R}^3 \setminus I, \tag{27}$$

where $p(s) = (0,0,s) \in I$. Note that v is the potential produced by a charge distribution on I, whose density linearly varies from 0 to 1. Consequently, we have $\Delta v = 0$ in $\mathbb{R}^3 \setminus I$, and in particular, $v \in C^\infty(\mathbb{R}^3 \setminus I)$. It is easy to compute

$$v(x) = |x-p(1)| - |x| + x_3\log(1 - x_3 + |x-p(1)|) - x_3\log(-x_3 + |x|). \tag{28}$$

We will be interested in the behaviour of $v(x)$ as $x \to 0$. First of all, since $-x_3 + |x| \geq 2|x_3|$ for $x_3 \leq 0$, if we send $x \to 0$ while keeping $x_3 \leq 0$, then $v(x) \to 1$. To study what happens when $x_3 > 0$, we write

$$v(x) = v_0(x) - x_3\log\left(|x_1|^2 + |x_2|^2\right), \tag{29}$$

with

$$v_0(x) = |x-p(1)| - |x| + x_3\log(1 - x_3 + |x-p(1)|) + x_3\log(x_3 + |x|). \tag{30}$$

The function v_0 is continuous in $\mathbb{R}^3 \setminus \{0, p(1)\}$ with $v_0(x) \to 1$ as $x \to 0$. Moreover, if we send $x \to 0$ in the region $|x_1|^2 + |x_2|^2 \geq |x_3|^n$ with some n, then we still have $v(x) \to 1$. On the other hand, if we send $x \to 0$ along a curve with $|x_1|^2 + |x_2|^2 = e^{-\alpha/x_3}$ for some constant $\alpha > 0$, then we have $v(x) \to 1 + \alpha$. We also note that because of the singularity at

$x_1 = x_2 = 0$ of the last term in (29), we see that $v(x) \to +\infty$ as x approaches $I \setminus \{0\}$. Now we define $\Omega = \{x : v(x) < 1+\alpha\} \cap B_1$ with a sufficiently large $\alpha > 0$. Then although $v(0)$ can be defined so that v is continuous on $\partial\Omega$, it is not possible to extend v to a function in $C(\overline{\Omega})$.

Next, consider the Dirichlet problem $\Delta u = 0$ in Ω, and $u = v$ on $\partial\Omega$. Let $M = \|u-v\|_{L^\infty(\Omega)}$, and for $\varepsilon > 0$, let $\Omega_\varepsilon = \Omega \setminus \overline{B}_\varepsilon$. Then the function

$$w(x) = \frac{M\varepsilon}{|x|} \pm (u(x) - v(x)), \tag{31}$$

satisfies $\Delta w = 0$ in Ω_ε and $w \geq 0$ on $\partial\Omega_\varepsilon$. By the minimum principle, we have $w \geq 0$ in Ω_ε, which means that

$$|u(x) - v(x)| \leq \frac{M\varepsilon}{|x|}, \qquad x \in \Omega_\varepsilon. \tag{32}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $u = v$. ⊘

If p is an isolated boundary point, i.e., if p is the only boundary point in a neighbourhood of it, then as Zaremba observed, one cannot specify a boundary condition at p because it would be a removable singularity for the harmonic function in the domain. It is shown by Osgood in 1900 that this is the only possible way for a boundary point of a *two dimensional* domain to be irregular.

Theorem 12 (Osgood 1900)**.** *Let $\Omega \subset \mathbb{R}^2$ be open and let $p \in \partial\Omega$ be contained in a component of $\mathbb{R}^2 \setminus \Omega$ which has more than one point (including p). Then p is regular.*

Proof. It will be convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and without loss of generality, to assume that $p = 0$. Let $w \in \mathbb{C}$ be another point so that both p and w are contained in the same connected component of $\mathbb{C} \setminus \Omega$. After a possible scaling, we can assume that $|w| > 1$. Moreover, since regularity is a local property, we can restrict attention to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, that is, we assume that $\Omega \subset \mathbb{D}$. Let $z_0 \in \Omega$ and consider a branch of logarithm near z_0. This branch can be extended to Ω as a single-valued function, for if it were not, there must exist a closed curve in Ω that goes around the origin. However, it is impossible because there is a connected component of $\mathbb{C} \setminus \Omega$ that contains 0 and w. Denoting the constructed branch by log, we claim that $\varphi(z) = \mathrm{Re}(\log z)^{-1}$ is a barrier. Since $\log z$ is a holomorphic function that vanishes nowhere in Ω, we have $\Delta\varphi = 0$ in Ω. Moreover, we have $\varphi(z) \to 0$ as $z \to 0$ and $\varphi < 0$ in Ω because $\mathrm{Re}(\log z) = \log |z|$ and $|z| < 1$ for $z \in \Omega$. This shows that φ is indeed a barrier at 0. □