FIRST ORDER EQUATIONS

TSOGTGEREL GANTUMUR

Abstract. After reviewing some fundamental results from calculus in Banach spaces, we give a justification to the method of characteristics for first order equations.

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1. Vector-valued calculus

Let $I \subset \mathbb{R}$ be an interval, and let X be a Banach space. We want to develop a basic calculus for functions $f : I \to X$. First of all, let us define the *uniform norm*

$$\|f\|_\infty = \sup_{t \in I} \|f(t)\|, \tag{1}$$

where $\|\cdot\|$ denotes the norm in X, and the corresponding space of *bounded functions*

$$B(I, X) = \{f : I \to X : \|f\|_\infty < \infty\}. \tag{2}$$

We denote by $C(I, X)$ the space of *continuous functions* on I having values in X. Then the space of *bounded continuous functions* is defined by

$$C_b(I, X) \equiv BC(I, X) = B(I, X) \cap C(I, X). \tag{3}$$

For $f \in C(I, X)$, the norm $\|f(t)\|$ is a continuous function of $t \in I$. Therefore if I is a closed interval we have $C(I, X) = C_b(I, X)$.

Exercise 1. Show that $C_b(I, X)$ is a Banach space under the norm $\|\cdot\|_\infty$.

If for a function $f : I \to X$ and $a \in I$, there is some $\lambda \in X$ such that

$$f(t) - f(a) = \lambda(t - a) + o(|t - a|), \qquad \text{as} \quad t \to a, \tag{4}$$

then we say that f is *differentiable at* $a \in I$, and write

$$f'(a) = \dot{f}(a) = \frac{\mathrm{d}f}{\mathrm{d}t}(a) = \lambda, \tag{5}$$

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which is called the *derivative of f at a*. In case a is an endpoint of I, the limit in (4) should be understood as a one-sided limit. If f is differentiable everywhere on I, then f' is clearly a function on I having values in X. So the following definitions make sense

$$\begin{aligned} C^k(I,X) &= \{f \in C(I,X) : f' \in C^{k-1}(I,X)\}, \\ C_b^k(I,X) &= \{f \in C_b(I,X) : f' \in C_b^{k-1}(I,X)\}. \end{aligned} \tag{6}$$

Now we want to introduce the Riemann integral. A *partition* of an interval $[a,b]$ is a sequence $p = \{t_i\}_{i=0}^n$ satisfying

$$a = t_0 < t_1 < \ldots < t_n = b, \tag{7}$$

and we say that p is *tagged* by $\xi = \{\xi_i\}_{i=1}^n$ if $\xi_i \in [t_{i-1}, t_i]$ for $i = 1, \ldots n$. Given a tagged partition (p,ξ), the *Riemann sum* of $f \in B([a,b],X)$ with respect to (p,ξ) is defined by

$$S_{p,\xi}(f) = \sum_{i=1}^n (t_i - t_{i-1}) f(\xi_i). \tag{8}$$

Then we say that f is *Riemann integrable on* $[a,b]$ if $S_{p,\xi}(f)$ has a limit in X as $|p| \to 0$, where $|p|$ is the *width*

$$|p| = \max_i |t_i - t_{i-1}|. \tag{9}$$

If f is integrable, we take its *Riemann integral* over $[a,b]$ to be

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{|p|\to 0} S_{p,\xi}(f). \tag{10}$$

We also have the convention

$$\int_b^a f(t)\,\mathrm{d}t = -\int_a^b f(t)\,\mathrm{d}t. \tag{11}$$

Since there is no risk of confusion we simply drop the adjective "Riemann" from integrability and integral. We have a simple criterion on integrability, in terms of the quantity

$$\operatorname{osc}(f,p) = \sum_{i=1}^n (t_i - t_{i-1}) \operatorname{osc}(f, [t_{i-1}, t_i]), \tag{12}$$

which could be called the *oscillation of f* over the partition p, where

$$\operatorname{osc}(f, [t_{i-1}, t_i]) = \sup_{\xi,\eta \in [t_{i-1}, t_i]} \|f(\xi) - f(\eta)\|, \tag{13}$$

is the oscillation of f over the interval $[t_{i-1}, t_i]$. Note that a function f is continuous at t if and only if $\operatorname{osc}(f,[r,s]) \to 0$ as $r \nearrow t$ and $s \searrow t$, and that if f is continuous, then the oscillation $\operatorname{osc}(f,[r,s])$ is a continuous function of r and s.

Lemma 1. *A function $f \in B([a,b],X)$ is integrable over $[a,b]$ if for any $\varepsilon > 0$ there is a partition p such that* $\operatorname{osc}(f,p) < \varepsilon$.

Proof. Note that if p' is a refinement of p, i.e., if $p \subset p'$, then $\operatorname{osc}(f,p') \le \operatorname{osc}(f,p)$ and

$$\|S_{p,\xi}(f) - S_{p',\xi'}(f)\| \le \sum_{i=1}^n (t_i - t_{i-1}) \operatorname{osc}(f, [t_{i-1}, t_i]) = \operatorname{osc}(f,p), \tag{14}$$

for any sets of tags ξ and ξ', respectively, of p and p'. Let $p_0, p_1, \ldots$ be a sequence of partitions with oscillations tending to 0. Replacing p_i by the common refinement $p_i \cup p_0$, we can assume that $p_i \subset p_{i+1}$ for all i. Then from (14) we see that the Riemann sums corresponding to the sequence $\{p_i\}$ converge to some element $x \in X$, independent of how the partitions are tagged. The convergence can be established first for some particular tagging of the sequence $\{p_i\}$, and then be extended to arbitrary tagging by (14). Now we need to show that as long as the width is small, any tagged partition gives rise to a Riemann sum that is close to x. Let (q,η) be a tagged partition with $|q|$ small. Let $q' = q \cup p_k$ with k large. We tag q' by η' such that η' coincides with η on the subintervals common to both q and q', meaning that

$$\|S_{q,\eta}(f) - S_{q',\eta'}(f)\| \leq \sum_I |q| \operatorname{osc}(f, I) \leq 2\#p_k|q|\|f\|_\infty, \tag{15}$$

where the sum is over all subintervals I of q satisfying $I \cap p_k \neq \varnothing$, and $\#p_k$ denotes the number of nodes in p_k. Given $\varepsilon > 0$, we choose k so large that $\|S_{p_k,\cdot}(f) - x\| < \varepsilon$. Then if $|q|$ is so small that the right hand side of (15) is less than ε, we get

$$\|S_{q,\eta}(f) - x\| \leq \|S_{q,\eta}(f) - S_{q',\eta'}(f)\| + \|S_{q',\eta'}(f) - x\| < 2\varepsilon, \tag{16}$$

establishing the claim. □

The following lemma produces a large class of integrable functions.

Corollary 2. *Functions in $C([a,b],X)$ are integrable.*

Proof. Since $C([a,b],X) \subset B([a,b],X)$, it suffices to produce a sequence of partitions with oscillations vanishing in the limit. The oscillation $\operatorname{osc}(f,[s,t])$ is a continuous function of (s,t) on the closed triangle $\{a \leq s \leq t \leq b\}$, and $\operatorname{osc}(f,[t,t]) = 0$. Hence the modulus of continuity $\omega(\delta) = \max_t \operatorname{osc}(f,[t,t+\delta])$ is a continuous function of δ with $\omega(0) = 0$. This means that for any given $\varepsilon > 0$, there exists a partition p of $[a,b]$ with $\operatorname{osc}(f,p) < \varepsilon$. □

Note that the proof shows that a continuous function on a closed interval is *uniformly continuous*, in the sense that $\omega(\delta) \to 0$ as $\delta \searrow 0$.

Obviously, integration is a linear operation. A few more simple properties are as follows.

Lemma 3. *If $f : [a,b] \to X$ is integrable, then we have the additivity*

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t, \qquad c \in [a,b], \tag{17}$$

and the bounds

$$\Big\| \int_a^b f(t)\,\mathrm{d}t \Big\| \leq (b-a)\|f\|_\infty, \qquad \textit{and} \qquad \Big\| \int_a^b f(t)\,\mathrm{d}t \Big\| \leq \int_a^b \|f(t)\|\,\mathrm{d}t. \tag{18}$$

Moreover, if $\{f_n\}$ is a sequence of integrable functions on $[a,b]$ converging uniformly to f in $[a,b]$, then f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t. \tag{19}$$

Proof. The additivity (17) and inequalities (18) are true because they are true when the integrals are replaced by the corresponding Riemann sums. For any partition p (with any tag), we have

$$\begin{aligned} \Big| \int_a^b f_n(t)\,\mathrm{d}t - S_p(f) \Big| &\leq \Big| \int_a^b f_n(t)\,\mathrm{d}t - S_p(f_n) \Big| + |S_p(f_n) - S_p(f)| \\ &\leq \Big| \int_a^b f_n(t)\,\mathrm{d}t - S_p(f_n) \Big| + (b-a)\|f_n - f\|_\infty, \end{aligned} \tag{20}$$

which implies that f is integrable and that the limit (19) holds. □

For X and Y Banach spaces, let $L(X,Y)$ be the space of bounded linear operators between X and Y. That is, the space $L(X,Y)$ consists of linear operators $A : X \to Y$, for which the *operator norm*

$$\|A\| = \sup_{x\in X} \frac{\|Ax\|_Y}{\|x\|_X}, \tag{21}$$

is finite.

Exercise 2. Show that $L(X,Y)$ is a Banach space under the operator norm.

Lemma 4. *If $A:[a,b]\to L(X,Y)$ is continuous, and $x\in X$ then we have*

$$\int_a^b A(t)x\,\mathrm{d}t = \left(\int_a^b A(t)\,\mathrm{d}t\right)x. \tag{22}$$

Similarly, if $f:[a,b]\to X$ is continuous, and $A\in L(X,Y)$ then we have

$$\int_a^b Af(t)\,\mathrm{d}t = A\left(\int_a^b f(t)\,\mathrm{d}t\right). \tag{23}$$

Proof. We only prove (22). By linearity, we have $S_p(Ax)=S_p(A)x$ for Riemann sums. The claim follows from the fact that $S_p(Ax)$ converges to the left hand side of (22) in Y, while $S_p(A)$ converges to the integral in right hand side of (22) in $L(X,Y)$. □

Theorem 5 (Fundamental theorem of calculus)**.** *a) If $u\in C^1([a,b],X)$ then*

$$u(b)-u(a)=\int_a^b u'(t)\,\mathrm{d}t. \tag{24}$$

b) If $f\in C([a,b],X)$ then the function

$$F(x)=\int_a^x f(t)\,\mathrm{d}t, \qquad x\in[a,b], \tag{25}$$

satisfies $F\in C^1([a,b],X)$ and $F'=f$ on $[a,b]$.

Proof. a) Consider the function

$$g(r,s)=u'(r)-\frac{u(s)-u(r)}{s-r}, \tag{26}$$

on the triangle $\{a\le r<s\le b\}$. This is continuous, and can be continuously extended to $\{r=s\}$ as $g(r,r)=0$. Hence

$$g(h)=\max_r \|g(r,r+h)\|, \tag{27}$$

is continuous for $h\ge 0$ and $g(0)=0$, which is to say that

$$\big\|u'(t)-\frac{u(t+h)-u(t)}{h}\big\|\le g(h), \tag{28}$$

uniformly in t.

Let us take a partition p of evenly spaced points in $[a,b]$, with each subinterval tagged at its left endpoint. Let $h=|p|$ be the length of each subinterval. Then we have

$$\begin{aligned}\|S_p(u')-[u(b)-u(a)]\| &= \big\|\sum_{i=1}^n hu'(t_{i-1})-\sum_{i=1}^n (u(t_i)-u(t_{i-1}))\big\| \\ &\le \sum_{i=1}^n h\big\|u'(t_i)-\frac{u(t_i)-u(t_{i-1})}{h}\big\| \\ &\le \sum_{i=1}^n hg(h)=g(h),\end{aligned} \tag{29}$$

which shows that $S_p(u')\to u(b)-u(a)$ as $h\to 0$. Since u' is integrable this limit must be equal to the integral.

b) By additivity, for any $a\le x<y\le b$ we have

$$F(y)-F(x)=\int_x^y f(t)\,\mathrm{d}t. \tag{30}$$

On the other hand, we have

$$\int_x^y f(x)\,\mathrm{d}t = (y-x)f(x). \tag{31}$$

Taking the difference of the two expressions, we infer[1]

$$\|F(y) - F(x) - (y-x)f(x)\| \leq \int_x^y \|f(t) - f(x)\|\,\mathrm{d}t \leq (y-x)\mathrm{osc}(f,[x,y]), \tag{32}$$

which shows that

$$\frac{F(y) - F(x)}{y-x} \to f(x) \qquad \text{as} \quad y \to x. \tag{33}$$

To treat the right endpoint b, we take $y = b$, replace $f(x)$ in (31) by $f(y)$, and send $x \to y$ in the end rather than $y \to x$. □

2. Banach's fixed point theorem

A *distance function*, or a *metric*, on a set M is a function $\rho : M \times M \to \mathbb{R}$ that is symmetric: $\rho(u,v) = \rho(v,u)$, nonnegative: $\rho(u,v) \geq 0$, nondegenerate: $\rho(u,v) = 0 \Leftrightarrow u = v$, and satisfies the triangle inequality: $\rho(u,v) \leq \rho(u,w) + \rho(w,v)$. Then a *metric space* is a set with a metric. If a sequence $\{u_n\}$ in M satisfies $\rho(u_n, u) \to 0$ as $n \to \infty$ for some $u \in M$, we say that the sequence *converges* to u, and write $u_n \to u$ in M. It is obvious that convergent sequences are *Cauchy*, meaning that $\rho(u_n, u_m) \to 0$ as $n, m \to \infty$. In general, however, Cauchy sequences do not have to converge in the space, as can be seen from, e.g, the example $M = \mathbb{Q}$ and $\rho(x,y) = |x-y|$. If the metric space M is such that every Cauchy sequence converges to an element of M, we call it a *complete metric space*. A mapping $\phi : M \to W$ between two metric spaces is called *continuous* if $u_n \to u$ in M implies $\phi(u_n) \to \phi(u)$ in W. With ϱ denoting the metric of W, if

$$\varrho(\phi(u), \phi(v)) \leq k\rho(u,v), \qquad u, v \in M, \tag{34}$$

with some constant $k \in \mathbb{R}$, then we say that ϕ is *Lipschitz continuous*. In this setting, ϕ is called a *nonexpansive* mapping if $k \leq 1$, and a *contraction* if $k < 1$.

Theorem 6. *Let M be a non-empty, complete metric space, and let $\phi : M \to M$ be a contraction. Then ϕ has a unique fixed point, i.e., there is a unique $u \in M$ such that $\phi(u) = u$.*

Proof. Uniqueness follows easily from nondegeneracy of the metric. For existence, starting with some $u_0 \in M$, define the sequence $\{u_n\}$ by $u_n = \phi(u_{n-1})$ for $n = 1, 2, \ldots$. Then this sequence is Cauchy, because

$$\rho(u_n, u_{n+1}) = \rho(\phi(u_{n-1}), \phi(u_n)) \leq k\rho(u_{n-1}, u_n) \leq \ldots \leq k^n \rho(u_0, u_1), \tag{35}$$

and so

$$\rho(u_n, u_m) \leq \rho(u_n, u_{n+1}) + \ldots + \rho(u_{m-1}, u_m) \leq (k^n + \ldots + k^{m-1})\rho(u_0, u_1) \leq \frac{k^n \rho(u_0, u_1)}{1-k}, \tag{36}$$

for $n < m$. Since M is complete, there is $u \in M$ such that $u_n \to u$, which is a good candidate for the fixed point we are looking for. Indeed, we have

$$\rho(u, \phi(u)) \leq \rho(u, u_n) + \rho(\phi(u_{n-1}), \phi(u)) \leq \rho(u, u_n) + k\rho(u_{n-1}, u) \to 0, \tag{37}$$

as $n \to \infty$, showing that $u = \phi(u)$. □

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[1] This argument was suggested by Joshua Lackman.

Proof due to Dick Palais. The argument (36) can be replaced by the following more symmetric one. For any $u, v \in M$, we have

$$\rho(u,v) \leq \rho(u,\phi(u)) + \rho(\phi(u),\phi(v)) + \rho(\phi(v),v) \leq \rho(u,\phi(u)) + k\rho(u,v) + \rho(\phi(v),v), \quad (38)$$

implying that

$$(1-k)\rho(u,v) \leq \rho(u,\phi(u)) + \rho(\phi(v),v). \quad (39)$$

Using this in combination with (35), we get

$$(1-k)\rho(u_n,u_m) \leq \rho(u_n,u_{n+1}) + \rho(u_{m+1},u_m) \leq (k^n + k^m)\rho(u_0,u_1), \quad (40)$$

hence $\{u_n\}$ is Cauchy. □

Remark 7. If the map ϕ depends on some parameter continuously, then the fixed point obtained by the preceding theorem also depends continuously on the parameter. This can be seen as follows. Let $u_s = \phi_s(u_s)$ and $u_t = \phi_t(u_t)$, with both ϕ_s and ϕ_t contractions. Then from the triangle inequality we have

$$\rho(u_s,u_t) \leq \rho(\phi_s(u_s),\phi_s(u_t)) + \rho(\phi_s(u_t),\phi_t(u_t)) \leq k\rho(u_s,u_t) + \rho(\phi_s(u_t),\phi_t(u_t)), \quad (41)$$

hence

$$(1-k)\rho(u_s,u_t) \leq \rho(\phi_s(u_t),\phi_t(u_t)), \quad (42)$$

meaning that if $\phi_s(u_t)$ and $\phi_t(u_t)$ are close then so are u_s and u_t.

Remark 8. The continuous dependence on parameters can be updated to differentiability if there is enough structure. Let us assume that M is a closed subset of a Banach space X, and let $\{\phi_t\}$ be a one-parameter family of contractions on M, with the contraction factor k bounded away from 1 uniformly in t. Suppose that $u_t = \phi_t(u_t)$ are the fixed points and that u_0 is a point interior to M. We also assume that $\phi_t(u)$ is differentiable at $\{t = 0, u = u_0\}$ in the sense that there exist $\lambda \in X$ and a bounded linear operator $\Lambda : X \to X$ such that

$$\phi_t(u_0 + tv) = \phi_0(u_0) + \lambda t + t\Lambda v + o(|t|), \quad (43)$$

as $t \to 0$. For $t \neq 0$ small, the difference quotients Δ_t satisfy

$$\Delta_t := \frac{u_t - u_0}{t} = \frac{\phi_t(u_0 + t\Delta_t) - \phi_0(u_0)}{t} =: \Phi_t(\Delta_t). \quad (44)$$

The map Φ_t is a contraction on its domain because

$$|t| \cdot \|\Phi_t(x) - \Phi_t(y)\| \leq \|\phi_t(u_0 + xt) - \phi_t(u_0 + yt)\| \leq k|t| \cdot \|x - y\|. \quad (45)$$

Moreover, from the differentiability condition we have

$$\Phi_t(x) \to \lambda + \Lambda x, \qquad \text{as} \quad t \to 0, \quad (46)$$

for any $x \in X$, so $\Phi_0(x) = \lambda + \Lambda x$ is a contraction by continuity. Hence there exists a fixed point $\Delta_0 \in X$ of Φ_0 with $\Delta_t \to \Delta_0$ as $t \to 0$, and of course Δ_0 is the derivative of u_t with respect to t at $t = 0$. We can continue in this manner. For instance, the continuity of the derivative Δ_0 with respect to some parameter would follow if λ and Λ depend continuously on that parameter.

3. Inverse function theorem

Let X and Z be Banach spaces, equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively, and let $U \subset X$ be an open set. Then a mapping $f : U \to Z$ is called *Fréchet differentiable* at $x \in U$ if

$$f(x+h) = f(x) + \Lambda h + o(\|h\|_X), \qquad \text{as} \quad X \ni h \to 0, \tag{47}$$

for some bounded linear operator $\Lambda : X \to Z$. We call $Df(x) = \Lambda$ if it exists, the *Fréchet derivative* of f at x. If $Df(x)$ exists for each $x \in U$, then Df can be considered as a map sending U to $L(X,Z)$, the space of bounded linear operators between X and Z. We can now define the space $C^1(U,Z)$, as the space of functions $f : U \to Z$ satisfying the condition that Df exists everywhere in U, and $Df : U \to L(X,Z)$ is continuous.

The following results are fundamental.

Theorem 9 (Inverse function theorem)**.** *Suppose that $Df \in C^1(U,Z)$, and that $Df(x)$ is invertible. Then there is an open neighbourhood V of x such that $f : V \to Z$ is injective, and that $f^{-1} \in C^1(f(V),X)$.*

Proof. Given $z \in Z$ close to $f(x)$, consider the map $\phi : U \to X$ defined by

$$\phi(y) = y + [Df(x)]^{-1}(z - f(y)), \tag{48}$$

which has the property that $\phi(y) = y$ if and only if $f(y) = z$.

If $y, y' \in U$ are close to x, then we have

$$\begin{aligned} \phi(y) - \phi(y') &= y - y' + [Df(x)]^{-1}(f(y') - f(y)) \\ &= [Df(x)]^{-1}(f(y') - f(y) + Df(x)(y - y')) \\ &= [Df(x)]^{-1}(Df(x) - Df(y))(y - y') + o(\|y - y'\|_X), \end{aligned} \tag{49}$$

where we have used the fact that

$$f(y') - f(y) = Df(y)(y' - y) + o(\|y - y'\|_X). \tag{50}$$

By the continuity of $y \mapsto Df(y)$ as a function having values in the space of bounded linear operators between X and Z, this shows that we can choose a closed neighbourhood $B \subset U$ of x, with the property that $\phi : B \to X$ is a contraction.

Now for $y \in B$, we have

$$\begin{aligned} \phi(y) - x &= [Df(x)]^{-1}(z - f(y) + Df(x)(y - x)) \\ &= [Df(x)]^{-1}(z - f(x) + f(x) + Df(x)(y - x) - f(y)) \\ &= [Df(x)]^{-1}(z - f(x)) + o(\|y - x\|_X), \end{aligned} \tag{51}$$

meaning that there is a closed neighbourhood $B' \subset B$ of x and an open neighbourhood E of $f(x)$, such that $\phi : B' \to B'$ as long as $z \in E$. Hence the Banach fixed point theorem implies that ϕ has a unique fixed point in B', that is, $f : B' \to E$ is invertible.

Finally, we look at the differentiability of $f^{-1} : E \to B'$. Let $y \in U$ be sufficiently close to x that $Df(y)$ is invertible. Then with $y' \in U$ sufficiently close to y, we have

$$f(y') = f(y) + Df(y)(y' - y) + o(\|y' - y\|_X), \tag{52}$$

and hence

$$[Df(y)]^{-1}(f(y') - f(y)) = y' - y + o(\|y' - y\|_X). \tag{53}$$

This implies the existence of a constant C such that

$$\|y' - y\|_X \leq C\|f(y') - f(y)\|_Z, \tag{54}$$

for all y' in a small neighbourhood of y. Taking this into account, and writing $z = f(y)$ and $z' = f(y')$, from (53) we infer

$$f^{-1}(z') = f^{-1}(z) + [Df(y)]^{-1}(z' - z) + o(\|z' - z\|_Z), \tag{55}$$

which means that $Df^{-1}(z)$ exists and is equal to $[Df(y)]^{-1}$. In particular, f^{-1} is continuous in a small neighbourhood of $f(x)$, and therefore $Df^{-1}(z) = [Df(f^{-1}(z))]^{-1}$ is continuous as a function of z in a small neighbourhood of $f(x)$. □

Corollary 10 (Implicit function theorem)**.** *Let X, Y and Z be Banach spaces, and with $A \subset X \times Y$ an open set, let $g \in C^1(A, Z)$. Moreover, assume that the point $(a, b) \in A$ has the property that $g(a, b) = 0$ and that $D_y g(a, b)$ is invertible, where $D_y g(x, y)$ is the Fréchet derivative of $y \mapsto g(x, y)$, with fixed x. Then there is an open set $U \subset X$ and a function $h \in C^1(U, Y)$ with $h(a) = b$, such that $g(x, h(x)) = 0$ for all $x \in U$.*

Proof. We apply the inverse function theorem to the function $f(x, y) = (x, g(x, y))$. □

There is a weaker and more easily accessible derivative, called Gâteaux derivative, that plays an important role in calculus of variations. This is simply the Banach space version of the directional derivative. We define the *Gâteaux differential of $f : U \to Z$ at $x \in U$ in the direction $a \in X$*, to be $Df(z, a) = g'(0)$ if the latter exists, where

$$g(t) = f(x + at), \tag{56}$$

is a function of $t \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$. Assuming that $Df(x, a)$ exists for all $a \in X$ and for all $x \in U$, the totality of all Gâteaux differentials of f defines a function $(x, a) \mapsto Df(x, a)$ on $U \times X$. There is no obvious *a priori* structure on this function, except to say that $Df(x, a)$ is homogeneous in a, i.e., $Df(x, at) = tDf(x, a)$ for any $t \in \mathbb{R}$.

If f is Gâteaux differentiable at x and the map $a \mapsto Df(x, a) : X \to Z$ is a bounded linear map, we call it the *Gâteaux derivative* of f at x. Obviously, Fréchet differentiability implies the existence of the Gâteaux derivative. In the other direction, we have the following result, which gives a practical way to get a handle on Fréchet derivatives.

Lemma 11. *Suppose that the Gâteaux differential of f exists at each $x \in U$ as a bounded linear map $A(x) : X \to Z$, and that $A : U \to L(X, Z)$ is continuous. Then f is Fréchet differentiable in U, with $Df = A$.*

Proof. We assume $0 \in U$ and will show that f is Fréchet differentiable at $0 \in U$. The continuity of $A : U \to B(X, Z)$ at 0 means that for any $\varepsilon > 0$, there is a small ball $B_\delta \subset U$ centred at 0, such that $x \in B_\delta$ implies

$$\|A(x)y - A(0)y\| \leq \varepsilon \|y\|, \tag{57}$$

for all $y \in X$. Pick $\varepsilon > 0$, let $x \in B_\delta$ with δ as above, and define $g(t) = f(xt)$. We compute

$$g'(t) = \lim_{\varepsilon \to 0} \frac{f(xt + x\varepsilon) - f(xt)}{\varepsilon} = Df(xt, x) = A(xt)x, \tag{58}$$

for $0 \leq t \leq 1$. We have

$$f(x) = g(1) = f(0) + \int_0^1 g'(t)\,\mathrm{d}t = f(0) + A(0)x + \int_0^1 [A(xt)x - A(0)x]\,\mathrm{d}t, \tag{59}$$

and hence

$$\|f(x) - f(0) - A(0)x\| \leq \sup_{0 \leq t \leq 1} \|A(xt)x - A(0)x\| \leq \varepsilon \|x\|, \tag{60}$$

showing that f is Fréchet differentiable at 0 with $Df(0) = A(0)$. Note that we could have set $f(0) = 0$ and $A(0) = 0$ in the beginning to simplify the formulas. □

Example 12. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We apply the preceding lemma to show that

$$f(u) = \int_\Omega |u|^p, \tag{61}$$

is Frećhet differentiable in the space $L^p(\Omega)$ with $2 < p < \infty$. First, we have to compute the Gâteaux derivative. Since $|u + t\psi|^p = |u|^p + tp|u|^{p-2}u\psi + o(t)$ pointwise, we expect the Gâteaux derivative to be

$$Df(u, \psi) = p \int_\Omega |u|^{p-2}u\psi. \tag{62}$$

We need to justify the limit when $\psi \in L^p(\Omega)$. From Taylor's theorem we have

$$|u + t\psi|^p = |u|^p + tp|u|^{p-2}u\psi + \tfrac{1}{2}t^2p(p-1)|u + st\psi|^{p-2}\psi^2, \tag{63}$$

where $0 \leq s \leq 1$ is a function on Ω. We can use the inequality

$$(a + b)^\alpha \leq \max\{1, 2^{\alpha-1}\}(a^\alpha + b^\alpha) \qquad \text{for} \qquad a, b \geq 0, \quad \alpha > 0, \tag{64}$$

to estimate

$$\tfrac{1}{2}t^2p(p-1)|u + st\psi|^{p-2}\psi^2 \leq C(t^2|u|^{p-2}\psi^2 + |t|^p|\psi|^p), \tag{65}$$

with some constant $C > 0$. This implies that

$$\begin{aligned} \Big|\int_\Omega \Big(\frac{|u + t\psi|^p - |u|^p}{t} - p|u|^{p-2}u\psi\Big)\Big| &\leq C\int_\Omega (|t||u|^{p-2}\psi^2 + |t|^{p-1}|\psi|^p) \\ &\leq C|t|\|u\|_p^{p-2}\|\psi\|_p^2 + C|t|^{p-1}\|\psi\|_p^p, \end{aligned} \tag{66}$$

proving that (62) is indeed the Gâteaux derivative of f. Here $\|\cdot\|_q$ denotes the L^q-norm, and in the last line we have used the Hölder inequality with the exponents $\frac{p}{p-2}$ and $\frac{p}{2}$.

For fixed $u \in L^p(\Omega)$, the map $\psi \mapsto Df(u, \psi)$ is clearly linear, and bounded on L^p since

$$|Df(u, \psi)| \leq p\|u\|_p^{p-1}\|\psi\|_p, \tag{67}$$

by the Hölder inequality with the exponents $\frac{p}{p-1}$ and p.

Now we will show the continuity of $u \mapsto Df(u, \cdot) : L^p \to (L^p)'$, with the latter space taken with its norm topology. For any constant $a > 0$, the function $g(x) = |x|^a x$ is continuously differentiable with

$$g'(x) = (a + 1)|x|^a, \tag{68}$$

implying that

$$\big||x|^a x - |y|^a y\big| \leq (a + 1)\Big|\int_x^y |t|^a \mathrm{d}t\Big| \leq (a + 1)\max\{|x|^a, |y|^a\}|x - y|. \tag{69}$$

Using this, we have

$$\begin{aligned} |f(u, \psi) - f(v, \psi)| &\leq p\int_\Omega \big||u|^{p-2}u - |v|^{p-2}v\big| \cdot |\psi| \\ &\leq p(p-1)\int_\Omega \big(|u|^{p-2} + |v|^{p-2}\big)\,|u - v| \cdot |\psi|. \end{aligned} \tag{70}$$

Finally, it follows from the Hölder inequality with the exponents $\frac{p}{p-2}$, p, and p that

$$|f(u, \psi) - f(v, \psi)| \leq p(p-1)\left(\|u\|_p^{p-2} + \|v\|_p^{p-2}\right)\|u - v\|_p \cdot \|\psi\|_p, \tag{71}$$

which establishes the claim.

4. Ordinary differential equations

Let X be a Banach space, and let $\Omega \subset X$ be an open set. Let $I \subset \mathbb{R}$ be an open interval with $0 \in I$, and let $f : \Omega \times I \to X$ be a continuous map. We consider the *initial value problem*

$$\begin{cases} u'(t) = f(u(t), t) & t \in I, \\ u(0) = x, \end{cases} \tag{72}$$

where $x \in \Omega$ is given, called the *initial datum*. If f does not depend on t explicitly, i.e., if $f(y, t) = f(y)$, then the initial value problem is said to be *autonomous*.

Definition 13. A *classical solution* of (72) is a function $u \in C^1(I, X)$ satisfying (72). On the other hand, a *strong solution* of (72) is a function $u \in C(I, X)$ satisfying

$$u(t) = x + \int_0^t f(u(s), s)\,\mathrm{d}s, \qquad t \in I. \tag{73}$$

Lemma 14. *A function $u \in C(I, X)$ is a strong solution of (72) if and only if it is a classical solution of (72).*

Proof. Suppose that $u \in C^1(I, X)$ is a classical solution. Then for any $0 \leq t \in I$, we have $u \in C^1([0, t], X)$, and hence by the fundamental theorem of calculus (Theorem 5a)

$$u(t) - x = u(t) - u(0) = \int_0^t u'(s)\,\mathrm{d}s = \int_0^t f(u(s), s)\,\mathrm{d}s. \tag{74}$$

Similarly, for any $0 \geq t \in I$, we have $u \in C^1([t, 0], X)$, and so

$$x - u(t) = u(0) - u(t) = \int_t^0 u'(s)\,\mathrm{d}s = -\int_0^t f(u(s), s)\,\mathrm{d}s. \tag{75}$$

Now suppose that $u \in C(I, X)$ is a strong solution. Then obviously $u(0) = x$, and since $[t \mapsto f(u(t), t)] \in C(I, X)$, Theorem 5b) gives $u \in C^1(I, X)$ and $u'(t) = f(u(t), t)$ for $t \in I$. $\square$

Now we want to establish local existence and uniqueness theorems for the initial value problem (72). To this end, we need the following notion.

Definition 15. A map $f : \Omega \times I \to X$ is called *locally Lipschitz in the first variable*, if for any $r > 0$ and any closed interval $J \subset I$, there is $C_{r,J} > 0$ such that

$$\|f(x, t) - f(y, t)\| \leq C_{r,J}\|x - y\|, \qquad x, y \in Q_r,\ t \in J, \tag{76}$$

where $Q_r = \{x \in \Omega : \|x\| \leq r,\ \mathrm{dist}(x, X \setminus \Omega) \geq \frac{1}{r}\}$.

Theorem 16 (Local existence, Picard-Lindelöf, or Cauchy-Lipschitz)**.** *Assume that f is locally Lipschitz in its first variable. Then for any $r > 0$ there exists an open interval $I \ni 0$ such that for initial data $x \in Q_r$, the initial value problem (72) has a strong solution $u \in C(I, X)$ that depends on x continuously.*

Proof. With $T > 0$ and $J = [-T, T]$, consider the map $\phi_x : C(J, X) \to C(J, X)$, defined by

$$\phi_x(u)(t) = x + \int_0^t f(u(s), s)\,\mathrm{d}s, \qquad t \in J. \tag{77}$$

Obviously, u is a strong solution on J iff $u = \phi_x(u)$. Our plan is to show that ϕ_x is a contraction on some closed subset of $C(J, X)$ so that we can apply the Banach fixed point theorem.

Suppose that $u, v \in U_R$, where $U_R = C(J, Q_R)$ for some $R > r$. Then we have

$$\|\phi_x(u) - \phi_x(v)\|_\infty \leq \Big| \int_0^t \|f(u(s), s) - f(v(s), s)\|\,\mathrm{d}s \Big| \leq T C_{R,J}\|u - v\|_\infty, \tag{78}$$

and similarly

$$\begin{aligned}\|\phi_x(u)-x\|_\infty \le \Big|\int_0^t \|f(u(s),s)\|\,\mathrm{d}s\Big| \le \Big|\int_0^t \|f(u(s),s)-f(z,s)\|\,\mathrm{d}s\Big| + \Big|\int_0^t \|f(y,s)\|\,\mathrm{d}s\Big| \\ \le TC_{R,J}\|u-z\|_\infty + T\|f(z,\cdot)\|_\infty \le 2TC_{R,J}R + T\|f(z,\cdot)\|_\infty,\end{aligned} \tag{79}$$

where $z \in Q_r$ is fixed independently of x.

We see that for any $R > r$, by choosing $T > 0$ small enough we can ensure that $\phi(u) \in U_R$ for $u \in U_R$, and that ϕ is a contraction on U_R. Note that the constant $C_{R,J}$ does not grow when we shrink the interval $J = [-T, T]$. Since U_R is a closed subset of a Banach space, it is a complete metric space (with the metric induced by the norm), and hence an application of the Banach fixed point theorem gives the existence of $u \in U_R$ satisfying $u = \phi(u)$.

Now suppose that $v \in U_R$ satisfies $v = \phi_y(v)$, where ϕ_y is given by (77) with x replaced by $y \in Q_r$. Then we have

$$\|u-v\|_\infty \le \|x-y\| + \|\phi_0(u)-\phi_0(v)\|_\infty \le \|x-y\| + k\|u-v\|_\infty, \tag{80}$$

with some $0 < k < 1$, where we have used $\phi_x(u) = x + \phi_0(u)$ and $\phi_y(v) = y + \phi_0(v)$. This implies that

$$\|u-v\|_\infty \le \frac{1}{1-k}\|x-y\|, \tag{81}$$

which is the desired continuity result. □

Note that in the preceding theorem we have constructed a local solution that depends on x continuously. If we try a bit harder, we can extract a local uniqueness result from this theorem, but we want to apply a more robust uniqueness argument. To this end, we need the following important inequality.

Lemma 17 (Gronwall's inequality)**.** *Let y and b be continuous functions on $[0, T]$ with $b \ge 0$, and let A be a real constant. Assume*

$$y(t) \le A + \int_0^t b(s)y(s)\,\mathrm{d}s, \qquad t \in [0, T]. \tag{82}$$

Then we have

$$y(t) \le A \exp \int_0^t b(s)\,\mathrm{d}s, \qquad t \in [0, T]. \tag{83}$$

Proof. Let

$$g(t) = A + \int_0^t b(s)y(s)\,\mathrm{d}s, \qquad \text{and} \qquad z(t) = g(t)\exp\big(-\int_0^t b(s)\,\mathrm{d}s\big). \tag{84}$$

Then we have

$$g'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(A + \int_0^t b(s)y(s)\,\mathrm{d}s\big) = b(t)y(t) \le b(t)\big(A + \int_0^t b(s)y(s)\,\mathrm{d}s\big) = b(t)g(t), \tag{85}$$

hence

$$z'(t) = g'(t)\exp\big(-\int_0^t b(s)\,\mathrm{d}s\big) - g(t)\exp\big(-\int_0^t b(s)\,\mathrm{d}s\big)b(t) \le 0. \tag{86}$$

This implies $z(t) \le A$ and

$$y(t) \le g(t) \le A \exp \int_0^t b(s)\,\mathrm{d}s, \tag{87}$$

which completes the proof. □

Theorem 18 (Uniqueness)**.** *Assume that f is locally Lipschitz in its first variable. If $I_1 \ni 0$ and $I_2 \ni 0$ are two intervals and $u_1 \in C(I_1, X)$ and $u_2 \in C(I_2, X)$ are two strong solutions of (72), then $u_1 = u_2$ on $I_1 \cap I_2$.*

Proof. Let $T > 0$ be such that $T \in I_1 \cap I_2$. Then for $t \in [0, T]$ we have

$$\|u_1(t) - u_2(t)\| = \Big\| \int_0^t (f(u_1(s), s) - f(u_2(s), s))\,\mathrm{d}s \Big\| \leq C_{R,[0,T]} \int_0^t \|u_1(s) - u_2(s)\|\,\mathrm{d}s, \quad (88)$$

where R is such that $u_1, u_2 \in C([0,T], Q_R)$. Now Gronwall's inequality applied to the function $y(t) = \|u_1(t) - u_2(t)\|$ gives $u_1 - u_2 = 0$ on $[0, T]$. The case $T < 0$ is the same because under the substitution $v(t) = u(-t)$, the problem $u' = f(u, t)$ becomes $v' = -f(v, -t)$. □

Definition 19. Let $u_1 \in C(I_1)$ and $u_2 \in C(I_2)$ be two solutions of (72) over $I_1 \ni 0$ and $I_2 \ni 0$, respectively. Then u_1 is called an *extension* of u_2 if $I_1 \supset I_2$ and $u_1|_{I_2} = u_2$.

Remark 20. Let $u_1 \in C(I_1)$ and $u_2 \in C(I_2)$ be two solutions of (72). Then the function $u \in C(I_1 \cup I_2)$ defined by

$$u(t) = \begin{cases} u_1(t), & t \in I_1, \\ u_2(t), & t \in I_2, \end{cases} \quad (89)$$

is an extension of each of u_1 and u_2. Note that u is well-defined thanks to Theorem 18. To see that the claim is true, if $t \in I_1 \cup I_2$ then $[0, t] \subset I_i$ for some $i \in \{1, 2\}$, and so we have

$$u(t) = u_i(t) = x + \int_0^t f(u_i(s), s)\,\mathrm{d}s = x + \int_0^t f(u(s), s)\,\mathrm{d}s, \quad (90)$$

meaning that u is a strong solution of (72) on $I_1 \cup I_2$.

Definition 21. A *maximal solution* is a solution that does not have any proper extension. The interval on which a maximal solution is defined is called a *maximal interval of existence.*

Theorem 22 (Maximal solutions)**.** *Assume that f is locally Lipschitz in its first variable. Then for any initial datum $x \in \Omega$, the initial value problem (72) has a maximal solution, and moreover the maximal solution is unique.*

Proof. Let $\{I_\alpha\}$ be the set of all intervals such that there is a solution $u_\alpha \in C(I_\alpha, X)$ of the initial value problem (72), where α runs over some index set. Let $I = I(x) = \bigcup_\alpha I_\alpha$, and define $u \in C(I, X)$ by

$$u(t) = u_{\alpha_t}(t), \qquad t \in I, \quad (91)$$

where $\alpha_t \in \{\alpha : t \in I_\alpha\}$ for each $t \in I$. Note that for each $t \in I$, the set $\{\alpha : t \in I_\alpha\}$ is nonempty, hence the existence of α_t satisfying $\alpha_t \in \{\alpha : t \in I_\alpha\}$ is guaranteed by the axiom of choice. Moreover, u is well-defined thanks to Theorem 18, and by applying the same argument as in (90), we see that u is a solution to (72). By construction, u cannot be extended to a larger interval of existence, hence its maximality.

For uniqueness, suppose that $u_1 \in C(I_1)$ and $u_2 \in C(I_2)$ are two maximal solutions of (72). Then by Remark 20 we can construct an extension u defined over $I_1 \cup I_2$. However, by maximality of u_1, we get $I_1 = I$, and similarly, $I_2 = I$. This gives $I_1 = I_2$, and hence by Theorem 18 we conclude $u_1 = u_2$. □

Theorem 23 (Blow-up criterion)**.** *The maximal interval of existence is necessarily open, i.e., it has the form $I = (a, b)$ for some $-\infty \leq a < 0 < b \leq \infty$. Moreover, if $b < \infty$, then $u(t)$ eventually escapes any Q_r as $t \nearrow b$, i.e., for any $r > 0$, there is $b' < b$ such that $u(t) \notin Q_r$ for all $t \in (b', b)$. Similarly, if $a > -\infty$, then $u(t)$ eventually escapes any Q_r as $t \searrow a$.*

Proof. Suppose that $I \cap [0, \infty) = [0, b]$ for some $b > 0$. Then $u(b) \in Q_r$ for some $r > 0$ by continuity. Hence the initial value problem

$$v'(t) = f(v(t), b + t), \qquad v(0) = u(b), \tag{92}$$

has a solution on some $[-\varepsilon, \varepsilon]$ with $\varepsilon > 0$. The function $t \mapsto u(b + t)$ satisfies the same equation for $t \le 0$ with the same initial datum, so $u(b+t) = v(t)$ for $-\varepsilon \le t \le 0$. This means that if we define

$$w(t) = \begin{cases} u(t) & t \le b, \\ v(t - b) & b < t \le b + \varepsilon, \end{cases} \tag{93}$$

it solves (72) on $[0, b + \varepsilon]$, which contradicts the maximality of b.

The second part can be proven by essentially the same argument. Suppose that $b < \infty$, and let $\{t_k\}$ be a sequence satisfying $t_k \nearrow b$. Assume $u(t_k) \in Q_r$ for some constant $r > 0$. Then by the local existence theorem, u can be continued up to time $t_k + T$ for each k, with $T > 0$ independent of k. Finally, choosing k large enough that $t_k + T > b$, we contradict the maximality of b. □

Example 24. Let $X = \mathbb{R}$, and consider the initial value problems $y' = y$ and $y' = y^2$, with $y(0) = g$. The solution of the first problem is $y(t) = ge^t$. Obviously, we have in this case $T = \infty$, i.e., the solution is *global in time*. Note that the solution is *unbounded*, because $y(t) \to \infty$ as $t \to \infty$. The second problem is solved by $y(t) = \frac{g}{1-gt}$, which *blows up* as $t \to T = \frac{1}{g}$, if $g \neq 0$. This is an example of a *finite time blow up*. More generally, nontrivial solutions to the initial value problem $y' = y^{1+\alpha}$ blow up in finite time as long as $\alpha > 0$. On the other hand, the problems $y' = y \log y$ and $y' = y(\log y) \log \log y$ have global in time solutions, even though they would grow like e^{e^t} and $e^{e^{e^t}}$, respectively. One can add more iterated logarithms without making the solution blow up in finite time, but any of those logarithms cannot be raised to a power greater than 1. For example, nontrivial solutions to the problems $y' = y(\log y)^{1+\alpha}$ and $y' = y(\log y)(\log \log y)^{1+\alpha}$ blow up in finite time, as long as $\alpha > 0$.

The preceding example indicates that if there a quadratic term in the equation, then there is a possibility of finite time blow up. However, if the unknown is a vector function, there can be many ways nonlinear terms does not cause blow up. The following example illustrates one such mechanism.

Example 25. Let $X = \mathbb{R}^3$, and consider the *Lorenz system*

$$\begin{cases} x' = \sigma(y - x) \\ y' = x(\rho - z) - y \\ z' = xy - \beta z \end{cases} \tag{94}$$

where σ, ρ, and β are positive parameters. We write $\xi = (x, y, z) \in \mathbb{R}^3$, and denote by $f(\xi) \in \mathbb{R}^3$ the vector in the right hand side of (94). Note that f is a *quadratic* expression in the components of ξ. The map $f : \mathbb{R}^3 \to \mathbb{R}^3$ is locally Lipschitz, since

$$|f(\xi_1) - f(\xi_2)| \le (\sigma + \rho + |y_1| + |z_1|)|x_1 - x_2| + (\sigma + 1 + |x_2|)|y_1 - y_2| + (\beta + |x_2|)|z_1 - z_2|, \tag{95}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, and $\xi_i = (x_i, y_i, z_i)$ for $i = 1, 2$. Therefore, given any $\xi_0 \in \mathbb{R}^3$, there is a unique maximal solution $\xi \in C^1((a, b), \mathbb{R}^3)$ of the initial value problem (94) with the initial condition $\xi(0) = \xi_0$. We want to prove that the solution is global in time, by using the blow-up criterion from Theorem 23. Suppose that $b < \infty$. Then for $t \in [0, b)$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\xi(t)|^2 &= 2(xx'+yy'+zz') = -2\sigma x^2 + 2(\sigma+\rho)xy - 2y^2 - 2\beta z^2 \\ &\leq (\rho-\sigma)x^2 + (\sigma+\rho-2)y^2 - 2\beta z^2 \leq (\sigma+\rho)|\xi(t)|^2,\end{aligned} \tag{96}$$

where the term $-xzy$ coming from the second row of (94) cancelled with the term xyz coming from the third row. Note that these two terms were the only potential sources that would give faster than quadratic growth in the right hand side (quadratic growth is harmless because we have $(|\xi|^2)'$ in the left hand side). Now an application of Gronwall's inequality gives

$$|\xi(t)|^2 \leq |\xi_0|^2 e^{(\sigma+\rho)t} \leq |\xi_0|^2 e^{(\sigma+\rho)b}. \tag{97}$$

The right hand side is clearly finite for any $b<\infty$, so it cannot happen that $|\xi(t)|\to\infty$ as $t\nearrow b$. By Theorem 23 this means that $b=\infty$. Similarly, we can get $a=-\infty$. To conclude, even though f is quadratic, since ξ is a vector, interactions between different directions result in a cancellation that prevents finite time blow up from happening.

Example 26. Let $K\in L^1(\mathbb{R})$, and let $A: C_b(\mathbb{R})\to C_b(\mathbb{R})$ be defined by

$$(Au)(x) = \int_{\mathbb{R}} K(y)u(x-y)\,\mathrm{d}y. \tag{98}$$

The map A is Lipschitz, since

$$\|Au-Av\|_\infty \leq \|K\|_{L^1(\mathbb{R})}\|u-v\|_\infty, \qquad u,v\in C_b(\mathbb{R}). \tag{99}$$

Therefore, given any $u_0\in C_b(\mathbb{R})$, there is a unique maximal solution $u\in C^1(I,C_b(\mathbb{R}))$ of the initial value problem

$$u' = Au, \qquad u(0)=u_0, \tag{100}$$

with $I=(a,b)$. We want to prove that $I=\mathbb{R}$, i.e., the global solvability of the problem, by using the blow-up criterion in Theorem 23. Suppose that (100) has a solution on $[0,b)$. Then for $t\in[0,b)$, we have

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + \int_0^t \|Au(s)\|_\infty\,\mathrm{d}s \leq \|u_0\|_\infty + \|K\|_{L^1}\int_0^t \|u(s)\|_\infty\,\mathrm{d}s, \tag{101}$$

which in combination with Gronwall's inequality implies that

$$\|u(t)\|_\infty \leq \|u_0\|_\infty \exp(t\|K\|_{L^1}) \leq \|u_0\|_\infty \exp(b\|K\|_{L^1}). \tag{102}$$

The right hand side is clearly finite for any $b<\infty$, so it cannot happen that $\|u(t)\|_\infty\to\infty$ as $t\nearrow b$. By Theorem 23 this means that $b=\infty$.

5. Flow maps

Given initial datum $x\in X$, let $u_x\in C(I_x,X)$ be the maximal solution of the initial value problem (72), where I_x is the maximal interval of existence corresponding to x. Then we define the *flow map* to be $\Phi_t(x)=u_x(t)$ for $t\in I_x$. Note that for any fixed t, Φ_t is defined on only a part of Ω, which is nonempty if t is small.

Theorem 27 (Group property)**.** *For autonomous systems, i.e., if the right hand side f does not depend on t explicitly, we have*

$$\Phi_0 = \mathrm{id}, \qquad \textit{and} \qquad \Phi_{t+s} = \Phi_t\circ\Phi_s, \tag{103}$$

whenever they make sense.

Proof. We will prove $\Phi_{t+s} = \Phi_t \circ \Phi_s$. Let $x \in X$ and let $t, s, t+s \in I_x$. Denote $u(t) = \Phi_{t+s}(x)$ and $v(t) = \Phi_t(\Phi_s(x))$. Then we have

$$u'(t) = \frac{\mathrm{d}\Phi_{t+s}(x)}{\mathrm{d}t} = \frac{\mathrm{d}\Phi_{t+s}(x)}{\mathrm{d}(t+s)} = f(\Phi_{t+s}(x)) = f(u(t)), \tag{104}$$

and $u(0) = \Phi_s(x)$. On the other hand, we have

$$v'(t) = \frac{\mathrm{d}\Phi_t(\Phi_s(x))}{\mathrm{d}t} = f(\Phi_t(\Phi_s(x))) = f(v(t)), \tag{105}$$

and $v(0) = \Phi_s(x)$. We see that u and v satisfy the same equations, with the same initial data. Hence by uniqueness, we have $u(t) = v(t)$. □

Theorem 28 (Continuity)**.** *For any $x \in \Omega$ and $t \in I_x$, there exists $\delta > 0$ such that Φ_t is defined on $B_\delta(x) = \{y \in X : \|y - x\| < \delta\}$, and $\Phi_t : B_\delta(x) \to \Omega$ is continuous. In fact, the maps $\Phi_\tau : B_\delta(x) \to \Omega$, for $\tau \in [0, t]$, are all Lipschitz continuous, with the Lipschitz constants bounded independently of τ.*

Proof. Without loss of generality, assuming that $t > 0$, let $r > 0$ be such that Q_r contains an open neighbourhood of the curve $\Phi_\tau(x)$, as τ ranges in $[0, t]$. Then the local existence theorem (Theorem 16) guarantees a solution at least on the time interval $[0, h]$ with some $h > 0$ independent of the initial condition, as long as the initial condition stays in Q_r. We subdivide the time interval $[0, t]$ into smaller subintervals $[0, t_1]$, $[t_1, t_2]$, and so on, until $[t_N, t]$, with the length of each subinterval not exceeding h. Let $x_i = \Phi_{t_i}(x)$ for all i. First, we choose $\delta_N > 0$ such that $\Phi_\tau(B_{\delta_N}(x_N)) \subset Q_r$ for all $\tau \in [0, h]$. This is possible because $\Phi_\tau(y)$ is defined for all $y \in Q_r$ and $\tau \in [0, h]$, and

$$\sup_{\tau \in [0,h]} \|\Phi_\tau(x) - \Phi_\tau(y)\| \leq c\|x - y\|, \tag{106}$$

for some constant $c > 0$. In the second step, we choose $\delta_{N-1} > 0$ such that $\Phi_\tau(B_{\delta_{N-1}}(x_{N-1})) \subset B_{\delta_N}(x_N)$ for all $\tau \in [0, h]$. We continue this process, until we reach the point x, where we choose $\delta > 0$ such that $\Phi_\tau(B_\delta(x)) \subset B_{\delta_1}(x_1)$ for all $\tau \in [0, h]$. Now, by construction, $\Phi_t : B_\delta(x) \to Q_r$ is the composition of finitely many Lipschitz continuous maps, which completes the proof. □

Remark 29. In the setting of the preceding theorem, $(\tau, y) \mapsto \Phi_\tau(y)$ is (jointly) continuous as a map sending $[0, t] \times B_\delta(x)$ into Ω, since $\Phi_\tau : B_\delta(x) \to \Omega$ is equicontinuous for all $\tau \in [0, t]$. That is, for any given $\varepsilon > 0$, we can choose $\rho > 0$ such that $\|\Phi_\tau(y_1) - \Phi_\tau(y_2)\| < \varepsilon$ for all $y_1, y_2 \in B_\delta(x)$ with $\|y_1 - y_2\| < \rho$ and for all $\tau \in [0, t]$.

Exercise 3. Consider the initial value problem

$$\begin{cases} u'(t) = f(u(t), t, \lambda) & t \in I, \\ u(0) = x, \end{cases} \tag{107}$$

where $\lambda \in B$ is a parameter, with $B = B_\varepsilon(0)$ an open ball in some Banach space Y. We assume that $f \in C(\Omega \times I \times B, X)$ is *locally Lipschitz in the first variable*, in the sense that for any $r > 0$, any closed interval $J \subset I$, and any $0 < \delta < \varepsilon$, there is $C_{r,J,\delta} > 0$ such that

$$\|f(x, t, \lambda) - f(y, t, \lambda)\| \leq C_{r,J,\delta}\|x - y\|, \qquad x, y \in Q_r,\, t \in J,\, \lambda \in B_\delta(0). \tag{108}$$

We have a maximal solution $u_{x,\lambda} \in C(I_{x,\lambda})$ for each $x \in \Omega$ and $\lambda \in B$. Then define the *solution map* Ψ_t by $\Psi_t(x, \lambda) = u_{x,\lambda}(t)$ for $t \in I_{x,\lambda}$. Similarly to the flow map, for any fixed t, Ψ_t is defined on only a part of $\Omega \times B$, which is nonempty if t is small. Prove the following: For any $x \in \Omega$ and $t \in I_{x,0}$, there exists $\delta > 0$ such that Ψ_t is defined on $B_\delta(x) \times B_\delta(0) \subset X \times Y$, and $\Psi_t : B_\delta(x) \times B_\delta(0) \to \Omega$ is continuous.

Theorem 30 (Differentiability)**.** *Assume that the system is autonomous, i.e., that the right hand side f does not depend on t explicitly. and let $f \in C^1(\Omega, X)$, with Df bounded on Q_r for any $r > 0$. Then for any $x \in \Omega$ and $t \in I_x$, there exists $\delta > 0$ such that $\Phi_t \in C^1(B_\delta(x), \Omega)$. Moreover, the Fréchet derivative $D\Phi_t(y) : X \to X$ is an invertible operator for all $y \in B_\delta(x)$.*

Proof. Let $\xi \in X$, and let

$$w_h(t) = \frac{\Phi_t(x + h\xi) - \Phi_t(x)}{h}, \tag{109}$$

for $h \in \mathbb{R}$ small and nonzero. Our first task is to show that the limit $w_h(t)$ as $h \to 0$ exists. To this end, we compute

$$w_h'(t) = \frac{1}{h}\Big(\frac{\mathrm{d}\Phi_t(x + he)}{\mathrm{d}t} - \frac{\mathrm{d}\Phi_t(x)}{\mathrm{d}t}\Big) = \frac{f(\Phi_t(x + he)) - f(\Phi_t(x))}{h}. \tag{110}$$

Thus upon introducing

$$F(y, t, h) = \frac{f(\Phi_t(x) + hy) - f(\Phi_t(x))}{h}, \tag{111}$$

for $y \in B_R$, $t \in [0, T]$, and $h \in (-\varepsilon, 0) \cup (0, \varepsilon)$, we can write

$$w_h'(t) = F(w_h(t), t, h). \tag{112}$$

Here $R > 0$ and $T \in I_x$ are arbitrary, and $\varepsilon = \varepsilon(R, T) > 0$ is chosen so that $\Phi_t(x) + hy \in Q_r$ for some small $r > 0$, and for all $(y, t, h) \in B_R \times [0, T] \times (-\varepsilon, \varepsilon)$. We would like to send $h \to 0$ in (112), and use continuity with respect to parameters (the preceding exercise) to imply that $w_h(t)$ has a limit as $h \to 0$.

The first step would be to define the right hand side $F(y, t, h)$ for points with $h = 0$, in such a way that the resulting function is (jointly) continuous in all three variables. We observe that $F(y, t, h) \to Df(\Phi_t(x))y$ as $h \to 0$, hence we set

$$F(y, t, 0) = Df(\Phi_t(x))y. \tag{113}$$

Continuity of F on $B_R \times [0, T] \times ((-\varepsilon, 0) \cup (0, \varepsilon))$ is clear from the definition (111). To see that it is continuous also at points with $h = 0$, we compute

$$\begin{aligned} F(y, t, h) - F(z, \tau, 0) &= \frac{f(\Phi_t(x) + hy) - f(\Phi_t(x))}{h} - Df(\Phi_\tau(x))z \\ &= Df(\Phi_t(x) + \theta hy)y - Df(\Phi_\tau(x))z, \end{aligned} \tag{114}$$

for h and y nonzero and for some $0 \leq \theta \leq 1$, where we have used the mean value theorem. After deriving it, we note that the final equality is true for $h = 0$ or $y = 0$ as well. The right hand side can be estimated as

$$\begin{aligned} &\|Df(\Phi_t(x) + \theta hy)y - Df(\Phi_\tau(x))z\| \\ &\qquad \leq \|Df(\Phi_t(x) + \theta hy) - Df(\Phi_\tau(x))\|\|y\| + \|Df(\Phi_\tau(x))\|\|y - z\|, \end{aligned} \tag{115}$$

which makes it clear that the difference $F(y, t, h) - F(z, \tau, 0)$ can be made arbitrarily small by choosing (y, t, h) close to $(z, \tau, 0)$.

Next, we look at Lipschitz continuity of F in its first variable. To this end, we note

$$\|F(y, t, 0) - F(z, t, 0)\| = \|Df(\Phi_t(x))(y - z)\| \leq \kappa\|y - z\|, \tag{116}$$

where $\kappa = \sup\limits_{t \in [0,T]} \|Df(\Phi_t(x))\|$. When $h \neq 0$, we have

$$F(y, t, h) - F(z, t, h) = \frac{f(\Phi_t(x) + hy) - f(\Phi_t(x) + hz)}{h} = Df(y^*)(y - z) \tag{117}$$

for some y^* lying on the line segment connecting $\Phi_t(x) + hy$ and $\Phi_t(x) + hz$, which gives

$$\|F(y, t, h) - F(z, t, h)\| \leq \|y - z\| \sup_{Q_r} \|Df\|, \tag{118}$$

confirming the Lipschitz continuity of F in its first variable.

Now we can take the limit $h \to 0$ in (112) with the initial datum $w_h(0) = \xi$, and conclude that $w_h(t)$ tends to $w_0(t)$, the latter being the solution of the same problem with $h = 0$, provided that $w_0(t)$ exists. Note that w_h exists on $[0, T]$ for $h \neq 0$ by definition (109). To see that w_0 exists on $[0, T]$, recall that w_0 is defined by

$$w_0'(t) = Df(\Phi_t(x))w_0(t), \qquad w_0(0) = \xi, \tag{119}$$

which is equivalent to

$$w_0(t) = \xi + \int_0^t Df(\Phi_\tau(x))w_0(\tau)\,\mathrm{d}\tau. \tag{120}$$

This implies that

$$\|w_0(t)\| \leq \|\xi\| + \int_0^t \|Df(\Phi_\tau(x))\|\|w_0(\tau)\|\,\mathrm{d}\tau \leq \|\xi\| + \kappa \int_0^t \|w_0(\tau)\|\,\mathrm{d}\tau, \tag{121}$$

leading to the bound

$$\|w_0(t)\| \leq \|\xi\| e^{\kappa t}, \tag{122}$$

by Gronwall's inequality. Hence the equation for w_0 is solvable at least on $[0, T]$, and we conclude that the Gâteaux *differential* of Φ_t exists at x along the direction ξ, which is given by $D\Phi_t(x, \xi) = w_0(t)$. It is easy to see from (120) that the correspondence $\xi \mapsto D\Phi_t(x, \xi)$ is linear, and in light of (122) we infer that $\xi \mapsto D\Phi_t(x, \xi)$ is a bounded linear operator in X, meaning that the Gâteaux *derivative* of Φ_t exists at x.

Let us denote the Gâteaux derivative of Φ_t at x by $J(t)$. In order to show that $J(t)$ is actually a Fréchet derivative, we will show that $J(t)$ is continuous as a function of x, and apply Lemma 11. From (120) and (122), we get

$$\|J(t)\xi - J(s)\xi\| = \Big\| \int_s^t Df(\Phi_\tau(x))w_0(\tau)\,\mathrm{d}\tau \Big\| \leq \kappa e^{\kappa T}|t - s|\|\xi\|, \tag{123}$$

for $s, t \in [0, T]$, which shows that $J : [0, T] \to L(X, X)$ is continuous. The problem (120) can be rewritten as

$$J(t)\xi = \xi + \int_0^t Df(\Phi_\tau(x))J(\tau)\xi\,\mathrm{d}\tau = \xi + \left(\int_0^t Df(\Phi_\tau(x))J(\tau)\,\mathrm{d}\tau \right) \xi, \tag{124}$$

for any $\xi \in X$, where we have used Lemma 4. Therefore, $J(t)$ solves the initial value problem

$$J'(t) = Df(\Phi_t(x))J(t), \qquad J(0) = \mathrm{id}. \tag{125}$$

If we consider this as an initial value problem with parameter x, then it is obvious that the right hand side $G(A, t, x) = Df(\Phi_t(x))A$ is Lipschitz continuous in the variable $A \in L(X, X)$, with the Lipschitz constant depending continuously on (t, x), and hence $J(t)$ depends on x continuously. We conclude that the Fréchet derivative $D\Phi_t(x)$ exists and is equal to $J(t)$.

To show that $D\Phi_t(x)$ is invertible, solve the initial value problem

$$K'(t) = -K(t)Df(\Phi_t(x)), \qquad K(0) = \mathrm{id}, \tag{126}$$

and note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(K(t)J(t)) &= K'(t)J(t) + K(t)J(t) \\ &= -K(t)Df(\Phi_t(x))J(t) + K(t)Df(\Phi_t(x))J(t) = 0, \end{aligned} \tag{127}$$

and that $K(0)J(0) = 0$. □

6. Semilinear equations

Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\alpha : \Omega \to \mathbb{R}^n$ be a vector field in Ω. We associate to α a differential operator $A = \sum_i \alpha_i \partial_i$. Then given a function $\beta : \Omega \times \mathbb{R} \to \mathbb{R}$, we consider the first order equation

$$Au(x) \equiv \sum_i \alpha_i(x)\partial_i u(x) = \beta(x, u(x)), \qquad x \in \Omega. \tag{128}$$

We will solve it by the *method of characteristics*.

Definition 31. A *parametric characteristic curve* of A is a map $\gamma \in C^1(I_\gamma, \Omega)$ with some open interval $I_\gamma \subset \mathbb{R}$, satisfying

$$\gamma'(t) = \alpha(\gamma(t)), \qquad t \in I_\gamma. \tag{129}$$

We call the image $[\gamma] = \gamma(I_\gamma) \subset \Omega$ a *characteristic curve* of A.

It is immediate from the definition that if γ is a parametric characteristic curve of A and if u is differentiable at the points of $[\gamma]$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t)) = \sum_i \partial_i u \cdot \gamma_i'(t) = \sum_i \partial_i u \cdot \alpha_i(\gamma(t)) = Au(\gamma(t)), \tag{130}$$

for all $t \in I_\gamma$. This transforms the equation (128) into an ODE problem.

Lemma 32. *Let $\alpha \in C^1(\Omega, \mathbb{R}^n)$ and let u be differentiable in Ω. Then (128) holds if and only if*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t)) = \beta(\gamma(t), u(\gamma(t))), \qquad t \in I_\gamma, \tag{131}$$

for every parametric characteristic curve γ of A.

Proof. If (128) holds and if γ is a parametric characteristic curve, then

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t)) = Au(\gamma(t)) = \beta(\gamma(t), u(\gamma(t))), \qquad t \in I_\gamma. \tag{132}$$

On the other hand, let $x \in \Omega$. Then by ODE theory there exists a parametric characteristic curve γ of A with $\gamma(0) = x$. Now if (131) holds then we have

$$Au(x) = Au(\gamma(t))|_{t=0} = \frac{\mathrm{d}u(\gamma(t))}{\mathrm{d}t}\Big|_{t=0} = \beta(\gamma(t), u(\gamma(t)))|_{t=0} = \beta(x, u(x)), \tag{133}$$

which completes the proof. □

The equation (131) means that once the value of u at some point of γ is fixed, then the value of u along γ is completely determined. Moreover, as long as u is differentiable, it can behave in an arbitrary fashion in directions transversal to the characteristic curves, since the lemma says that all (128) requires is (131). So we expect that the space of all solutions of (128) could be parameterized by a space of functions on a surface transversal to the characteristic curves. This leads us to the *Cauchy problem*: We are given a differentiable surface $\Gamma \subset \Omega$ and a function $g : \Gamma \to \mathbb{R}$, and consider the problem

$$\begin{cases} Au(x) = \beta(x, u(x)) & x \in \Omega, \\ u(\xi) = g(\xi) & \xi \in \Gamma. \end{cases} \tag{134}$$

We assume that Γ is *noncharacteristic*, meaning that $\alpha(\xi)$ is not tangent to Γ at any $\xi \in \Gamma$. The surface Γ is called the *Cauchy surface*, and g the *Cauchy data*.

For $\xi \in \Gamma$, let γ_ξ be the parametric characteristic curve with $\gamma_\xi(0) = \xi$, and let I_ξ be its maximal interval of existence (in Ω). Then we solve

$$\begin{cases} v_\xi'(t) = \beta(\gamma_\xi(t), v_\xi(t)), \\ v_\xi(0) = g(\xi). \end{cases} \tag{135}$$

It is possible that the maximal interval of existence of v_ξ is smaller than I_ξ. Let us call it $J_\xi \subset I_\xi$. We expect that the solution u should be given by the prescription

$$u(\gamma_\xi(t)) = v_\xi(t), \tag{136}$$

as ξ varies over Γ and t runs over J_ξ. However, there are some obstacles to our program:

- It may happen that one characteristic curve crosses the Cauchy surface Γ multiple times. In this case, there will be $x \in \Omega$ such that $x = \gamma_\xi(t) = \gamma_{\xi'}(t')$ with $\xi \neq \xi'$, and hence unless $g(\xi)$ is compatible with $g(\xi')$, there can exist no solutions.
- It may happen that some of the characteristic curves do not cross Γ. In this case there will be $x \in \Omega$ that is not reachable by any characteristic curve starting at Γ, and hence one cannot control the value of u at x. This leads to nonuniqueness.

In terms of the mapping $\varphi(\xi, t) = \gamma_\xi(t)$ that maps some region of the (ξ, t)-space into Ω, the first obstacle can be described as noninjectivity, while the second one is about nonsurjectivity. The following lemma says that in a certain sense those are the only obstacles.

Lemma 33. *Let Γ be a C^1 surface, and let g and α be C^1 functions. Let $\Sigma \subset \Gamma \times \mathbb{R}$ be an open set, such that $(\xi, t) \in \Sigma$ implies $t \in J_\xi$. Suppose that the mapping $\varphi : \Sigma \to \Omega$, defined by $\varphi(\xi, t) = \gamma_\xi(t)$, is invertible and that the inverse $(\xi, t) := \varphi^{-1} : \varphi(\Sigma) \to \Sigma$ is C^1. Then the function $u(x) = v_{\xi(x)}(t(x))$ for $x \in \varphi(\Sigma)$ solves the Cauchy problem (134).*

Proof. If $\eta \in \Gamma$ then $\xi(\eta) = \eta$ and $t(\eta) = 0$, so $u(\eta) = v_\eta(0) = g(\eta)$. If $x \in \varphi(\Sigma)$, then by assumption there is a unique pair (ξ, t) with $\xi \in \Gamma$ and $t \in J_\xi$. Moreover, by ODE uniqueness theory any parametric characteristic curve going through x agrees with γ_ξ, up to a time translation. Hence Lemma 32 in combination with (135) guarantees that u solves the Cauchy problem, provided that u is differentiable in Ω. But u is C^1, as follows from ODE theory, since $\xi(x)$ and $t(x)$ are both C^1 functions by assumption. □

If we have local C^1-invertibility, then global injectivity would guarantee global C^1-invertibility. While in general it is hard to say anything about global injectivity, local C^1-invertibility can be approached through the inverse function theorem.

Theorem 34. *Let Γ be a noncharacteristic C^1 surface, and let g and α be C^1 functions.*

a) Let $\xi \in \Gamma$. Then there exists a neighbourhood $U \subset \Omega$ of ξ, such that the Cauchy problem (134) with Ω replaced by U and Γ replaced by $\Gamma \cap U$, has a unique solution.

b) Let $\Sigma \subset \Gamma \times \mathbb{R}$ be an open set, such that $(\xi, t) \in \Sigma$ implies $t \in J_\xi$. Suppose that the mapping $\varphi : \Sigma \to \Omega$, defined by $\varphi(\xi, t) = \gamma_\xi(t)$, is injective, and let $(\xi, t) := \varphi^{-1} : U \to \Sigma$ be its inverse on $U = \varphi(\Sigma)$. Then the function $u(x) = v_{\xi(x)}(t(x))$ for $x \in U$ solves the Cauchy problem (134), and it is unique in U.

Proof. First of all, from ODE existence theory it is clear that there exists an open neighbourhood of $\Gamma \times \{0\}$ in $\Gamma \times \mathbb{R}$, on which φ is well defined. Let (ξ, t) be a point in that neighbourhood, and let us compute the derivative of φ at (ξ, t). Note that $\varphi(\xi, t) = \Phi_t(\xi)$ for $\xi \in \Gamma$, where Φ_t is the flow map of the vector field α. We have $\Phi_{t+s}(\xi) = \Phi_t(\Phi_s(\xi))$ for all small s. Differentiating this with respect to t gives

$$\frac{\partial}{\partial t}\Phi_{t+s}(\xi) = \frac{\partial}{\partial s}\Phi_{t+s}(\xi) = D\Phi_t(\Phi_s(\xi))\frac{\partial}{\partial s}\Phi_s(\xi), \tag{137}$$

and putting $s = 0$, we get

$$\frac{\partial}{\partial t}\Phi_t(\xi) = D\Phi_t(\xi)\alpha(\xi). \tag{138}$$

It follows that

$$\Phi_{t+s}(\xi + \eta) = D\Phi_t(\xi)\eta + D\Phi_t(\xi)\alpha(\xi)s + o(|\eta| + |s|), \tag{139}$$

for small $\eta \in \mathbb{R}^n$ and $s \in \mathbb{R}$, in particular showing that $\varphi \in C^1$. Moreover, by restricting $\xi \in \Gamma$ and $\eta \in T_\xi\Gamma$, the latter the tangent space of Γ at ξ, it tells us that

$$D\varphi(\xi, t)(\eta, s) = D\Phi_t(\xi)(\eta + \alpha(\xi)s), \tag{140}$$

with which we mean the derivative $D\varphi$ at (ξ, t) applied to the vector (η, s). Since $D\Phi_t(\xi)$ is invertible, the invertibility of $D\varphi(\xi, t)$ boils down to checking if one can recover the pair (η, s) uniquely from $\eta+\alpha(\xi)s$. But this is guaranteed by the noncharacteristic condition $\alpha(\xi) \notin T_\xi\Gamma$.

Part a) is established, since we can conclude that there is a neighbourhood of $(\xi, 0)$ in $\Gamma\times\mathbb{R}$ such that the mapping φ on that neighbourhood is continuously differentiable with $D\varphi(\xi, 0)$ invertible. Then the inverse function theorem would finish the job.

Part b) is also done, because we have local invertibility of φ by the inverse function theorem, which is updated to global invertibility by the injectivity assumption. □

Remark 35. When can we take $\Sigma = \bigcup_{\xi\in\Gamma}\{\xi\} \times J_\xi$, i.e., is it possible to include in U all points in Ω that are reachable by characteristics starting at Γ? This would be possible if characteristics starting at Γ do not return to Γ again. Indeed, if $\varphi(\xi, t) = \varphi(\eta, s)$, from ODE uniqueness theory we would have $\eta = \varphi(\xi, t-s)$. One case where characteristic are guaranteed not to return to the Cauchy surface is the case of *evolution equations*, which read

$$\partial_n u(x) + \sum_{i=1}^{n-1} \alpha_i(x)\partial_i u(x) = \beta(x, u(x)), \qquad x \in \Omega, \tag{141}$$

with the Cauchy surface $\Gamma = \Omega \cap \{x_n = 0\}$. The coordinate x_n is interpreted as time, and since $\alpha_n \equiv 1$, we see that for any parametric characteristic curve, we have $\gamma_n(t) = t$. Hence $\gamma(t) \in \Gamma$ necessarily implies $t = 0$.

7. Quasilinear and fully nonlinear equations

By allowing the vector field α to depend on the function value u as well, we arrive at the following *quasilinear equation*

$$\sum_i \alpha_i(x, u(x))\partial_i u(x) = \alpha_{n+1}(x, u(x)), \qquad x \in \Omega, \tag{142}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$ is considered as a vector field defined on some domain $\tilde{\Omega} \subset \mathbb{R}^{n+1}$. Now it is no longer possible to solve for the characteristics and the solution value separately, as in (129) and (135). Instead, we need to solve them simultaneously, resulting in curves that live in the graph of the solution. A *graph characteristic* of α is a map $\gamma \in C^1(I_\gamma, \tilde{\Omega})$ with some open interval $I_\gamma \subset \mathbb{R}$, satisfying

$$\gamma'(t) = \alpha(\gamma(t)), \qquad t \in I_\gamma. \tag{143}$$

Exercise 4. Let $\alpha \in C^1(\tilde{\Omega}, \mathbb{R}^{n+1})$, and let u be differentiable in Ω. Suppose that the graph of u is a subset of $\tilde{\Omega}$. Then u is a solution of (142) if and only if every graph characteristic that starts at a point on the graph of u stays in the graph of u for at least a short time.

Given a Cauchy surface $\Gamma \subset \Omega$ and initial datum $g : \Gamma \to \mathbb{R}$, the *Cauchy problem* takes the form

$$\begin{cases} \sum_i \alpha_i(x, u(x))\partial_i u(x) = \beta(x, u(x)) & x \in \Omega, \\ u(\xi) = g(\xi) & \xi \in \Gamma, \end{cases} \tag{144}$$

and *noncharacteristicity* of Γ reads as the vector $(\alpha_1(\xi, g(\xi)), \ldots, \alpha_n(\xi, g(\xi))) \in \mathbb{R}^n$ being not tangent to Γ at any $\xi \in \Gamma$.

In addition to the difficulties we had for semilinear equations with regard to global solvability, we encounter here a new obstacle that is caused by potential multi-valuedness of the function defined by the graph characteristics. Let us illustrate this phenomenon by *Burgers' equation*

$$\partial_t u + u\partial_x u = 0. \tag{145}$$

Assume that the initial datum $g \in C^1(\mathbb{R})$ is given. We can take the t coordinate as the parameter in the characteristic equations, since the equation for t would be $t' = 1$. Hence the characteristic equations are

$$\begin{cases} x'(t) = z(t), \\ z'(t) = 0, \end{cases} \tag{146}$$

where we decomposed the graph characteristics γ as $\gamma = (x, z)$. The initial conditions for the graph characteristic starting at $\xi \in \mathbb{R}$ are $x(0) = \xi$ and $z(0) = g(\xi)$. This is easily solvable and the solution is $x_\xi(t) = \xi + g(\xi)t$ and $z_\xi(t) = g(\xi)$. So the graph characteristics are straight lines orthogonal to the u-axis, with slopes in the xt-plane equal to their u-coordinate, and the solution can be written implicitly as $u(\xi + g(\xi)t) = g(\xi)$ wherever it is well-defined. Let $U(t) = \{(x, u(x, t)) : x \in \mathbb{R}\}$ be the graph of u at the time moment t. Then from the implicit formula we see that $U(t) = A(t)U(0)$, where $A(t)$ is the linear transformation of the (x, u) plane given by

$$A(t)(x, u) = (x + ut, u). \tag{147}$$

It shows that the points with higher u-coordinates move to the right with faster rate than the points with lower u-coordinates. Thus if g has a region where it decreases, as t grows the graph of g would become increasingly steep in that region, and eventually the upper part would go past the lower part of the graph, making itself a non-graph. This phenomenon is called *wave breaking* or *shock formation*.

Exercise 5. Let Γ be a C^1 surface, and let g and α be C^1 functions. Assume that Γ is noncharacteristic, and assume that the graph of g is in $\tilde{\Omega}$. Then there exists an open neighbourhood of Γ on which there exists a unique C^1 solution of the Cauchy problem (144).

The most general form of a first order equation is

$$F(x, u(x), \nabla u(x)) = 0, \qquad x \in \Omega, \tag{148}$$

where F is a C^1 function defined on a domain $\tilde{\Omega}$ of $\mathbb{R}^{2n+1}$. Interestingly, this can also be solved by a method of characteristics, where the characteristics are now curves in $\mathbb{R}^{2n+1}$. Let us label the arguments of F by $x \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $p \in \mathbb{R}^n$, so that, for example, $\partial_z F$ means the derivative of $F(x, z, p)$ with respect to z, and $\nabla_p F$ is the vector consisting of the derivatives of $F(x, z, p)$ with respect to $p_1, \ldots, p_n$. Suppose that u solves (148), and that $x = x(t)$ is a curve in Ω. Let $z(t) = u(x(t))$ and $p(t) = \nabla u(x(t))$. We want to derive equations for the curve $(x(t), z(t), p(t))$. First, note that

$$z'(t) = \nabla u(x(t)) \cdot x'(t) = p(t) \cdot x'(t). \tag{149}$$

Next, differentiating (148) with respect to x_i gives

$$0 = \frac{\partial F}{\partial x_i} = \partial_i F + \partial_z F \cdot \partial_i u + \nabla_p F \cdot \partial_i \nabla u, \tag{150}$$

and we have

$$p_i'(t) = \partial_i \nabla u(x(t)) \cdot x'(t) = \partial_i \nabla u \cdot \nabla_p F = -\partial_i F - \partial_z F \cdot \partial_i u = -\partial_i F - \partial_z F \cdot p_i, \tag{151}$$

where we have made the choice $x'(t) = \nabla_p F$. Collecting the relevant equations, we conclude

$$\begin{cases} x' = \nabla_p F \\ z' = p \cdot \nabla_p F \\ p' = -\nabla_x F - p\,\partial_z F. \end{cases} \tag{152}$$

Now supposing that a Cauchy surface $\Gamma \subset \Omega$ and initial datum $g \in C^1(\Gamma)$ are given, we attempt to solve the Cauchy problem of (148) with the initial condition $u|_\Gamma = g$ in the following manner. For each $\xi \in \Gamma$, we solve the characteristic equations (152) with the initial data $x(0) = \xi$, $z(0) = g(\xi)$, and $p(0) = h(\xi)$, the latter given implicitly by

$$\tau \cdot \nabla g = h(\xi) \cdot \tau \quad \text{for all} \quad \tau \in T_\xi \Gamma, \qquad \text{and} \qquad F(\xi, g(\xi), h(\xi)) = 0. \tag{153}$$

The first condition fixes the component of $h(\xi)$ that is tangential to Γ, while the second condition is supposed to give the normal component. At this level of generality, however, the second equation may not be solvable for $h(\xi)$, or even if it is solvable, the solution may not be unique. So we simply assume that it is solvable, and for each ξ, one of the solutions is chosen, which we denote by $h(\xi)$. Given such data, *noncharacteristicity* of Γ is expressed as

$$\nabla_p F(\xi, g(\xi), h(\xi)) \notin T_\xi \Gamma. \tag{154}$$

Under the noncharacteristic condition, one can prove a local existence result for the Cauchy problem of (148) similar to the semilinear and quasilinear cases. We refer to Evans §3.2.4 for a detailed proof, and end this section with some examples.

Example 36. Consider the *eikonal equation*

$$|\nabla u|^2 = 1. \tag{155}$$

We take $F(x, z, p) = \frac{1}{2}(|p|^2 - 1)$, and so $\nabla_p F = p$, $\nabla_x F = 0$, and $\partial_z F = 0$. The characteristic equations are

$$x' = p, \qquad z' = 1, \qquad p' = 0. \tag{156}$$

Given an initial datum g on a Cauchy surface Γ, the initial data for the characteristic curve starting at $\xi \in \Gamma$ are $x(0) = \xi$, $z(0) = g(\xi)$, and $p(0) = \nabla_\Gamma g(\xi) + \nu$ with $|p(0)| = 1$, where $\nu \perp T_\xi \Gamma$. We see that if $|\nabla_\Gamma g(\xi)| \geq 1$ then there is no solution, and if $|\nabla_\Gamma g(\xi)| < 1$ there are two possible choices for $p(0)$ depending on the direction of ν.

Example 37. An important class of first order equations is *Hamilton-Jacobi equations*, that are of the form

$$\partial_t u + H(x, t, \nabla u) = 0, \tag{157}$$

where H is a function of $2n + 1$ variables, and t is singled out as a time coordinate. To compare it with the general form (148), we introduce $\tilde{x} = (x, t)$ and $\tilde{p} = (p, \tau)$. With these variables, we can write $F(\tilde{x}, \tilde{p}) = H(x, t, p) + \tau$, and we have

$$\nabla_{\tilde{p}} F = (\nabla_p H, 1), \qquad \nabla_{\tilde{x}} F = (\nabla_x H, \partial_t H), \qquad \text{and} \qquad \partial_z F = 0. \tag{158}$$

We can use t as the parameter in the characteristic curves, since its equation would be $t' = 1$. Moreover, we can eliminate τ from the characteristic equations because $\tau = \partial_t u = -H$ by the equation. The end result is

$$\begin{cases} x' = \nabla_p H \\ p' = -\nabla_x H \\ z' = p \cdot \nabla_p H - H. \end{cases} \tag{159}$$

Given an initial datum g on $\{t = 0\}$, the initial data for the characteristic curve starting at $\xi \in \mathbb{R}^n \times \{0\}$ are $x(0) = \xi$, $p(0) = \nabla_x g(\xi)$, and $z(0) = g(\xi)$.