# MATH 580 ASSIGNMENT 5

DUE MONDAY DECEMBER 1

1. Let $\Omega \subset \mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n : x_n > 0\}$ be a domain, and let $\Sigma = \{x \in \partial\Omega : x_n = 0\}$ be a nonempty open subset of the hyperplane $\partial\mathbb{R}^n_+ \equiv \{x_n = 0\}$. Prove the following.
   a) Suppose that $u \in C^2(\Omega) \cap C(\Omega \cup \Sigma)$ is harmonic in $\Omega$ and $u = 0$ on $\Sigma$. Denote by $x^*$ the reflection $(x_1, \ldots, x_{n-1}, -x_n)$ of $x = (x_1, \ldots, x_{n-1}, x_n)$, and let
   $$\tilde{\Omega} = \Omega \cup \Sigma \cup \{x^* : x \in \Omega\}.$$
   Then the function $\tilde{u} \in C(\tilde{\Omega})$ defined by $\tilde{u} = u$ in $\Omega \cup \Sigma$ and $\tilde{u}(x^*) = -u(x)$ for $x \in \Omega$ is harmonic in $\tilde{\Omega}$. This result is known as the *Schwarz reflection principle.*
   b) The Cauchy problem for the Laplace equation $\Delta u = 0$ with the Cauchy data $u = 0$ and $\partial_n u = g$ on the hyperplane $\{x_n = 0\}$ has no solution in any neighbourhood of $0 \in \mathbb{R}^n$, if $g$ is not analytic at $0 \in \mathbb{R}^{n-1}$.
   c) A bounded harmonic function in $\mathbb{R}^n_+$ with $u = 0$ on $\{x_n = 0\}$ is identically zero.
2. Let $u \in C^2(\mathbb{R}^n)$ be an entire harmonic function in $\mathbb{R}^n$. Prove the following.
   a) If $u \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$ then $u \equiv 0$.
   b) Any tangent hyperplane to the graph of $u$ intersects the graph more than once.
   c) If $u$ satisfies $u(x) \geq -C(1 + |x|)^m$ for some constants $C$ and $m \in \mathbb{N}$, then $u$ is a polynomial of degree less or equal to $m$.
3. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in C^2(\Omega)$ be a nonconstant harmonic function. Show that $u$ cannot have a *local* maximum in $\Omega$. (Note that the strong maximum principle as usually stated rules out only *global* maximums.)
4. We say $u \in C(\Omega)$ is *subharmonic* in $\Omega$ if for each $y \in \Omega$ there exists $r^* > 0$ such that
   $$u(y) \leq \frac{1}{|B_r|} \int_{B_r(y)} u, \qquad \forall r \in (0, r^*). \tag{$*$}$$
   Prove the following statements.
   (a) A function $u \in C^2(\Omega)$ is subharmonic in $\Omega$ iff $\Delta u \geq 0$ in $\Omega$.
   (b) A function $u \in C(\Omega)$ is subharmonic in $\Omega$ iff for any closed ball $B \subset \Omega$ and any harmonic function $v$ in a neighbourhood of $\bar{B}$, $u \leq v$ on $\partial B$ implies $u \leq v$ in $B$.
   (c) We get an equivalent definition of subharmonic functions, if we replace the average over the ball $B_r(y)$ in $(*)$ by the average over the sphere $\partial B_r(y)$.
   (d) A function subharmonic in $\mathbb{R}^2$ and bounded from above must be constant. Is this statement true in $\mathbb{R}^n$ for $n \geq 3$?
5. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$.

*Date*: November 17, 2014.

(a) Show that if the Dirichlet problem in $\Omega$ is solvable for any boundary condition $g \in C(\partial\Omega)$, then each boundary point $z \in \partial\Omega$ admits a barrier.

(b) Why is regularity of a boundary point a local property? In other words, if $z \in \partial\Omega$ is regular, and if $\Omega'$ is a domain that coincides with $\Omega$ in a neighbourhood of $z$ (hence in particular $z \in \partial\Omega'$), then is $z$ also regular as a point on $\partial\Omega'$?