MATH 580 ASSIGNMENT 4

DUE MONDAY NOVEMBER 17

1. Show that in $\mathbb{R}^3$, the wave propagators form a one parameter group of linear operators. That is, defining the operator $W(t)$ for each $t \in \mathbb{R}$ as the operator that sends the pair (φ, ψ) of initial data to the pair $(u(\cdot, t), \partial_t u(\cdot, t))$, where u is the solution of the wave equation in $\mathbb{R}^3$ with the initial data $(u(\cdot, 0), \partial_t u(\cdot, 0)) = (\varphi, \psi)$, show that
$$W(s + t) = W(s)W(t), \qquad s, t \in \mathbb{R}.$$
You can make reasonable growth and regularity assumptions on the initial data.
2. Let u be a sufficiently smooth function satisfying
$$\Box u := \partial_t^2 u - \sum_{j,k=1}^{n} g_{jk} \partial_j \partial_k u = 0, \tag{$*$}$$
where (g_{jk}) is a real symmetric positive definite $n \times n$ matrix whose entries depend smoothly on $x \in \mathbb{R}^n$. Let
$$c^2 := \sup \sum_{j,k=1}^{n} g_{jk}(x) \xi_j \xi_k < \infty,$$
where the supremum is taken over all $x \in \mathbb{R}^n$ and $|\xi| = 1$. Suppose that
$$u(x, 0) = \partial_t u(x, 0) = 0, \qquad \text{for} \quad x \in \Omega,$$
where $\Omega \subset \mathbb{R}^n$ is an open set. Then show that $u(x, t) = 0$ whenever
$$c|t| < \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) \equiv \inf_{y \in \mathbb{R}^n \setminus \Omega} |x - y|.$$
3. In this exercise, we will construct high frequency asymptotics for the variable coefficient wave equation $(*)$.
   (a) Assuming the form
   $$u(x, t) = a(x, t) e^{i\psi(x,t)/\varepsilon}, \qquad a = a_0 + \frac{\varepsilon}{i} a_1 + \ldots + (\frac{\varepsilon}{i})^k a_k,$$
   derive the eikonal equation for the phase function ψ, and the transport equations for the amplitudes a_j.
   (b) Write down canonical equations for the method of characteristics to solve the eikonal equation with initial data $\psi(x, 0) = \varphi(x)$.

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(c) By requiring that we use t as the parameter in the canonical equations, derive an eikonal equation for φ. Alternatively, assume the form $\psi(x,t) = t - \varphi(x)$ and derive an eikonal equation for φ. Write down the transport equations for the amplitudes a_j in terms of φ.

(d) Write down canonical equations for the eikonal equation for φ. Show that they are equivalent to the *geodesic equation*

$$\begin{aligned}\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} &= g_{i\ell}\left(\tfrac{1}{2}\partial_\ell b_{jk} - \partial_j b_{k\ell}\right)\frac{\mathrm{d}x_j}{\mathrm{d}t}\frac{\mathrm{d}x_k}{\mathrm{d}t} \\ &\equiv \frac{1}{2} g_{i\ell}\left(\partial_\ell b_{jk} - \partial_j b_{k\ell} - \partial_k b_{j\ell}\right)\frac{\mathrm{d}x_j}{\mathrm{d}t}\frac{\mathrm{d}x_k}{\mathrm{d}t},\end{aligned}$$

where the repeated indices are summed over, and the matrix $[b_{ik}]$ is the inverse of the matrix $[g_{j\ell}]$, pointwise in $\mathbb{R}^n$.

4. Consider the function $v(x,t) = \frac{x}{t}E(x,t)$ for $x \in \mathbb{R}$ and $t > 0$, where

$$E(x,t) = \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{|x|^2}{4t}} \qquad (x \in \mathbb{R},\ t > 0),$$

is the heat kernel of $\mathbb{R}$. Show that $\partial_t v = \Delta v$ in $\mathbb{R} \times (0,\infty)$, and that $v(x,t) \to 0$ as $t \to 0^+$ for each fixed $x \in \mathbb{R}$. How do we reconcile this with Tychonov's uniqueness theorem?

5. Let $g \in C(\mathbb{R}^n)$ be a function satisfying

$$|g(x)| \leq M e^{\alpha |x|^2} \qquad (x \in \mathbb{R}^n),$$

where M and α are constants. Show that

$$u(x,t) = \int_{\mathbb{R}^n} E(x-y,t) g(y)\,\mathrm{d}y,$$

satisfies the heat equation

$$\partial_t u - \Delta u = 0, \qquad \text{in} \quad \mathbb{R}^n \times (0,T),$$

for some $T > 0$, and $u(\cdot,t) \to g$ locally uniformly in $\mathbb{R}^n$ as $t \to 0^+$. Can you take T arbitrarily large? Here

$$E(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \qquad (x \in \mathbb{R}^n,\ t > 0),$$

is the heat kernel of $\mathbb{R}^n$.

6. With $\Omega \subset \mathbb{R}^n$ a bounded open set, consider the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta u + au & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u = g & \text{on} \quad \Omega \times \{0\}, \end{cases}$$

where $g \in C(\bar{\Omega})$ and $a \in L^\infty(\Omega \times (0,\infty))$ are given functions. Assume the existence of a solution u in the class $C^2(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$.

(a) Show that the solution is unique in the same class.

(b) Assuming $a \equiv 0$, show that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le (4\pi t)^{-\frac{n}{2}} \|g\|_{L^1(\Omega)}, \qquad \text{for all} \quad t > 0.$$

(c) Show that there exists $c > 0$ with the property that if $\|a\|_\infty \le c$ then the L^2-norm of $u(\cdot,t)$ decays exponentially in time. You can use the following *Friedrich's inequality*: There is a constant C such that

$$\|v\|_{L^2(\Omega)} \le C \|\nabla v\|_{L^2(\Omega)},$$

for all $v \in C^1(\Omega) \cap C(\bar{\Omega})$ with $v|_{\partial\Omega} = 0$.