MATH 580 ASSIGNMENT 3

DUE MONDAY OCTOBER 27

1. Present a detailed proof of existence of a unique local analytic solution to the Cauchy problem for the Poisson equation $\Delta u = f$ in $\mathbb{R}^n$ with the Cauchy data $(u, \partial_n u)$ specified on $\Gamma = \{x_n = 0\}$, given the Cauchy-Kovalevskaya theorem for first order linear systems. (We obviously need to assume that f and the specified Cauchy data are analytic.)
2. In this exercise, we will prove the existence of local isothermal coordinates in two dimensions, under analyticity assumptions. Consider the second order linear operator
$$A = E(x,y)\partial_x^2 + 2F(x,y)\partial_x\partial_y + G(x,y)\partial_y^2,$$
where E, F, and G are real analytic functions in a neighbourhood of $0 \in \mathbb{R}^2$, and the matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is positive definite in the same neighbourhood. We want to find a neighbourhood $U \subset \mathbb{R}^2$ of 0, and a mapping $\phi = (u, v) : U \to \mathbb{R}^2$ such that
$$A = \psi\partial_u^2 + \psi\partial_v^2,$$
in the new coordinate system (u, v), with some positive function $\psi : U \to \mathbb{R}$.
   (a) Show that the aforementioned requirement is equivalent to the differential system
$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{pmatrix} = \begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix}. \qquad (*)$$
   (b) Show that $(*)$ is satisfied in a neighbourhood of 0 for some function $\psi > 0$, if
$$\begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \frac{1}{W}\begin{pmatrix} F & G \\ -E & -F \end{pmatrix}\begin{pmatrix} \partial_x v \\ \partial_y v \end{pmatrix}, \qquad (**)$$
   holds in a neighbourhood of 0, where $W = \sqrt{EG - F^2}$. What is ψ thus obtained?
   (c) Show that $(**)$ admits a solution $\phi = (u, v)$ in a neighbourhood U of 0. Make sure that ϕ defines a genuine coordinate system in U.

   Historical note: This result was proved by Gauss in 1822. The analyticity assumption was removed by Korn and Lichtenstein around 1915.
3. Consider the linear operator
$$A = \sum_{|\alpha| \le q} a_\alpha \partial^\alpha,$$
where $a_\alpha \in C^\infty(\Omega)$, with $\Omega \subset \mathbb{R}^n$ open. Let $\phi : \Omega \to \mathbb{R}^n$ be a diffeomorphism onto $\tilde{\Omega} = \phi(\Omega)$. We denote the generic point in Ω by x, and the generic point in $\tilde{\Omega}$ by y. Let

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b_α be the coefficients of A in the y-coordinates, i.e.,

$$\sum_{|\alpha|\leq q} a_\alpha(x)\partial_x^\alpha u(x) = \sum_{|\alpha|\leq q} b_\alpha(y)\partial_y^\alpha u(y), \qquad u \in C^\infty(\Omega),$$

where $y = \phi(x)$, and for simplicity of notation, we use $u(y)$ to mean the push-forward $u(\phi^{-1}(y))$. Show that the characteristic form is coordinate invariant, in the sense that

$$\sum_{|\alpha|=q} a_\alpha(x)\xi^\alpha = \sum_{|\alpha|=q} b_\alpha(y)\eta^\alpha, \qquad x \in \Omega,\ \eta \in \mathbb{R}^n,$$

where $y = \phi(x)$ and $\xi_j = \partial_j\phi_k(x)\eta_k$, with summation over k assumed in the latter. We see that "ξ transforms like a gradient", so ξ is really a cotangent vector at x.

4. Let Γ be an analytic hypersurface in $\mathbb{R}^n$, meaning that for each $x \in \Gamma$, there exists a neighbourhood $U \subset \mathbb{R}^n$ of x, and an analytic function $\phi : U \to \mathbb{R}$ with $\nabla\phi \neq 0$ on Γ, such that $\Gamma \cap U = \phi^{-1}(\{0\})$. Let $\Omega \subset \mathbb{R}^n$ be an open set containing Γ, and let $X : \Omega \to \mathbb{R}^n$ be an analytic vector field, satisfying $X(x) \notin T_x\Gamma$ for every $x \in \Gamma$. We consider the linear differential equation

$$Au \equiv \sum_{|\alpha|\leq q} a_\alpha \partial^\alpha u = f, \tag{$\dagger$}$$

where a_α ($|\alpha| \leq q$) and f are analytic functions in Ω, and we assume that Γ is nowhere characteristic for A. Prove that there exists an open set $\omega \subset \mathbb{R}^n$ containing Γ, such that there exists a unique analytic solution $u : \omega \to \mathbb{R}$ to the Cauchy problem for ($\dagger$) with the Cauchy data

$$X^k u = g_k, \quad k = 0, \ldots, q-1, \qquad \text{on } \Gamma,$$

where g_k ($k = 1, \ldots, q-1$) are analytic functions in Ω. Note that if we denote the components of X by $X_1, \ldots X_n$, the differential operator X^k is understood to be

$$X^k = (X_1\partial_1 + \ldots + X_n\partial_n)^k.$$

You can assume that the Cauchy-Kovalevskaya theorem for linear equations with flat Cauchy surfaces, as we proved in class, is known.

5. Give an example of an analytic function ϕ for which the initial value problem

$$u_t = u_{xx}, \qquad u(x,0) = \phi(x),$$

does not admit any solution that is analytic at $(x,t) = (0,0)$.

6. Consider the initial value problem

$$u_t + u_{xx} = 0, \qquad u(x,0) = \phi(x),$$

for the backward heat equation. For given $\varepsilon > 0$ and an integer $k > 0$, construct an initial datum ϕ such that

$$\|\phi\|_\infty + \ldots + \|\phi^{(k)}\|_\infty < \varepsilon,$$

and

$$\|u(\cdot,\varepsilon)\|_\infty > \frac{1}{\varepsilon}.$$