MATH 580 ASSIGNMENT 2

DUE WEDNESDAY OCTOBER 8

1. Consider the following *inhomogeneous linear transport problem*

$$\partial_t u(x,t) = \sum_{i=1}^{n} \alpha_i(x,t)\partial_i u(x,t) + \beta(x,t)u(x,t) + f(x,t), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

with the initial datum

$$u(x,0) = g(x), \qquad x \in \mathbb{R}^n.$$

We assume that all α_i, β, and f are C^1 functions of $n+1$ variables, and that g is a C^1 function of n variables. We also assume that

$$|\alpha(x,t)| \leq 1, \qquad |\beta(x,t)| \leq 1, \qquad |f(x,t)| \leq 1, \qquad (x,t) \in \mathbb{R}^{n+1},$$

where $|\alpha|$ is the Euclidean norm of the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$. Prove the following.

(a) There exists a unique solution $u \in C^1(\mathbb{R}^{n+1})$ to the aforementioned Cauchy problem.

(b) The solution satisfies the estimate

$$|u(x,t)| \leq \left(\|g\|_{L^\infty(B_{|t|}(x))} + |t| \right) e^{|t|}, \qquad (x,t) \in \mathbb{R}^{n+1},$$

where $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$.

(c) In addition, assume that $g \in L^2(\mathbb{R}^n)$,

$$\|f(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq 1, \qquad t \in \mathbb{R},$$

and

$$|\nabla \cdot \alpha(x,t)| \equiv |\textstyle\sum_i \partial_i \alpha_i(x,t)| \leq 1, \qquad (x,t) \in \mathbb{R}^{n+1}.$$

Then $u(\cdot,t) \in L^2(\mathbb{R}^n)$ for any $t \in \mathbb{R}$, and

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \left(\|g\|_{L^2(\mathbb{R}^n)} + \sqrt{|t|} \right) e^{2|t|}, \qquad t \in \mathbb{R}.$$

2. Consider the following *quasilinear equation*

$$\sum_i \alpha_i(x,u(x))\partial_i u(x) = \alpha_{n+1}(x,u(x)), \qquad x \in \Omega, \qquad (*)$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$ is a vector field defined on some domain $\tilde{\Omega} \subset \mathbb{R}^{n+1}$. A *graph characteristic* of α is a map $\gamma \in C^1(I_\gamma, \tilde{\Omega})$ with some open interval $I_\gamma \subset \mathbb{R}$, satisfying

$$\gamma'(t) = \alpha(\gamma(t)), \qquad t \in I_\gamma.$$

Prove the following.

Date: Fall 2014.

(a) Let $\alpha \in C^1(\tilde{\Omega}, \mathbb{R}^{n+1})$, and let u be differentiable in Ω. Suppose that the graph of u, $\mathrm{graph}(u) = \{(x, u(x)) : x \in \Omega\}$, is a subset of $\tilde{\Omega}$. Then u is a solution of $(*)$ if and only if every graph characteristic that starts at a point on the graph of u stays in the graph of u for at least a short time.

(b) Let $\Gamma \subset \Omega$ be a C^1 surface, and let $g \in C^1(\Gamma)$ and $\alpha \in C^1(\tilde{\Omega}, \mathbb{R}^{n+1})$. Let $\Omega \times \{0\} \subset \tilde{\Omega}$ and let the graph of g be in $\tilde{\Omega}$. Assume that Γ is *noncharacteristic*, that is, the vector $(\alpha_1(\xi, g(\xi)), \ldots, \alpha_n(\xi, g(\xi))) \in \mathbb{R}^n$ is not tangent to Γ at any $\xi \in \Gamma$. Then there exists an open set $U \subset \Omega$ containing Γ such that there exists a unique $C^1(U)$ solution of the Cauchy problem for $(*)$ with the initial datum $u(\xi) = g(\xi)$ for $\xi \in \Gamma$.

3. (a) Find the region of $\mathbb{R}^2$ in which the power series $\sum_n x_1^n x_2^{n!}$ is absolutely convergent.
   (b) The *domain of convergence* of a power series is the interior of the region in which the series converges absolutely. Exhibit a two-variable real power series whose domain of convergence is the unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$.
   (c) Show that if $\Omega \subset \mathbb{R}^2$ is a domain of convergence of some power series centred at 0, then Ω is reflection symmetric with respect to the coordinate axes, and $\{(\log x_1, \log x_2) : x \in \Omega,\ x_1 > 0,\ x_2 > 0\}$ is a convex domain.
4. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets, and let $u \in C^\omega(U, \mathbb{R}^m)$ and $v \in C^\omega(V, \mathbb{R}^k)$, with the range of u contained in V. Show that $v \circ u \in C^\omega(U, \mathbb{R}^k)$.
5. Consider the Laplace equation $\Delta u = 0$ on the unit disk, given in polar coordinates by $\mathbb{D} = \{(r, \theta) : r < 1\}$. Specify the Cauchy data

$$u(1, \theta) = f(\theta), \qquad \partial_r u(1, \theta) = g(\theta),$$

   where f and g are 2π-periodic real analytic functions. Then show that a real analytic solution exists in a neighbourhood of the circle $\partial\mathbb{D}$. Investigate what happens to the solution as $r \to 0$ and $r \to \infty$, if f and g are of the form

$$a_0 + \sum_{n=1}^{m} a_n \cos n\theta + b_n \sin n\theta,$$

   i.e., trigonometric polynomials.
6. For each of the following cases, determine the characteristic cones and characteristic surfaces.
   a) Wave equation with wave speed $c > 0$: $u_{xx} + u_{yy} = c^{-2} u_{tt}$.
   b) Tricomi-type equation: $u_{xx} + y u_{yy} = 0$.
   c) Ultrahyperbolic “wave” equation: $u_{xx} + u_{yy} = u_{zz} + u_{tt}$.