MATH 580 ASSIGNMENT 1

DUE MONDAY SEPTEMBER 22

1. Let $I \subset \mathbb{R}$ be an interval, and let $n \in \mathbb{N}$. Show that
$$C_b(I, \mathbb{R}^n) = \{u \in C(I, \mathbb{R}^n) : \|u\|_\infty \equiv \sup_{x \in I} |u(x)| < \infty\},$$
is a Banach space under the norm $\|\cdot\|_\infty$. Show that
$$U_R(v) = \{u \in C_b(I, \mathbb{R}^n) : \|u - v\|_\infty \leq R\},$$
where $v \in C_b(I, \mathbb{R}^n)$ and $R > 0$ are given, is a closed subset of $C_b(I, \mathbb{R}^n)$.
2. Consider the initial value problem
$$\begin{cases} u_\lambda'(t) = f(u_\lambda(t), \lambda), \\ u_\lambda(0) = x, \end{cases}$$
where $x \in \mathbb{R}^n$ is initial data, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuous function, such that $f(\cdot, \lambda) : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz for each fixed $\lambda \in \mathbb{R}$. We know that for each fixed $\lambda \in \mathbb{R}$, there is a unique maximal solution $u_\lambda \in C(I_\lambda, \mathbb{R}^n)$ with some interval $I_\lambda \ni 0$. Prove that u_λ depends continuously on the parameter λ, in the sense that for each $\lambda \in \mathbb{R}$ and each $t \in I_\lambda$, there exists $\delta > 0$ such that $t \in I_\mu$ for all $\mu \in (\lambda - \delta, \lambda + \delta)$, and that $u_\mu(t)$ is a continuous function of $\mu \in (\lambda - \delta, \lambda + \delta)$.
3. Prove that the *Rössler system*
$$\begin{cases} x' = -y - z \\ y' = x + ay \\ z' = b + z(x - c) \end{cases}$$
where a, b, and c are positive parameters, is globally well-posed, meaning that for any given initial data, a unique solution exists for all time $t \in \mathbb{R}$, which depends continuously on the initial data.
4. Solve
$$xu_x + 2yu_y + u_z = 3u, \qquad u(x, y, 0) = g(x, y).$$
5. Show that if $\beta \in \mathbb{R}$ and $u \in C^1(\mathbb{R}^2)$ is a solution of $u_t + \beta u_x = 0$, then
$$\{(x, t) : u \in C^k \text{ on a neighbourhood of } (x, t)\},$$
is a union of rays.

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6. Consider the equation

$$xu_x + (y - x^2)u_y = 0, \tag{$*$}$$

in the region $\{(x, y) : x > 0,\ y > 0\}$, where u is understood to be a function of (x, y).

(a) Determine and sketch the characteristics of $(*)$.

(b) For each of the following three initial conditions, verify if the corresponding initial value problem for $(*)$ has a solution, and if the solution is uniquely determined. In case there exists a unique solution, give an explicit expression for the solution.

$$\text{(i)} \quad u(x, 1) = x^3, \qquad \text{(ii)} \quad u(x, x) = x^3, \qquad \text{(iii)} \quad u(x, x^2) = x^3.$$

To clarify, in each case, x is a variable that takes positive real values, therefore specifying the initial condition on a curve in $\{(x, y) : x > 0,\ y > 0\}$.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.