MATH 580 FALL 2013 PRACTICE PROBLEMS

DECEMBER 3, 2013

1. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in C^2(\Omega)$ be a nonconstant harmonic function. Show that u cannot have a local maximum in Ω.
2. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in H^1_{\mathrm{loc}}(\Omega)$ satisfy
$$\int_\Omega \nabla u \cdot \nabla \varphi = 0, \qquad \text{for all} \quad \varphi \in \mathscr{D}(\Omega).$$
Show that $u \in C^\omega(\Omega)$ and that u is harmonic.
3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the *Friedrichs inequality*
$$\int_\Omega |u|^2 \leq C \int_\Omega |\nabla u|^2,$$
that holds for all $u \in H^1_0(\Omega)$, and for some constant $C = C(\Omega) > 0$.
   a) Characterize the best constant C via an eigenvalue problem.
   b) Compute the best constant for the cube $\Omega = (0,1)^n$.
4. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set having the H^1 extension property.
   a) Show that the first Neumann eigenvalue of Ω is $\lambda_1 = 0$, and the dimension of the eigenspace corresponding to this eigenvalue (i.e., the multiplicity of λ_1) is equal to the number of connected components of Ω.
   b) Show that Ω has finitely many connected components.