POISSON'S EQUATION

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Abstract. In these notes we will study the Poisson equation, that is the inhomogeneous version of the Laplace equation. Our starting point is the variational method, which can handle various boundary conditions and variable coefficients without any difficulty.

Contents

1. Variational method

Consider the following *Poisson problem*

$$\begin{cases} \Delta u + f = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and f is a given function. We begin by reformulating the problem in the Sobolev space setting as follows. Assume $f \in L^2(\Omega)$, and consider the problem of finding $u \in H^1_0(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \qquad \text{for all } v \in H^1_0(\Omega). \tag{2}$$

Note that the homogeneous boundary condition is reflected in the requirement $u \in H^1_0(\Omega)$. We call (2) the *weak formulation* (or the *variational formulation*) of (1).

Exercise 1. Show that if $u \in C^2(\Omega)$ satisfies (2) then $\Delta u + f = 0$ almost everywhere in Ω.

To get some insight on (2), we write it as

$$a(u, v) = F(v) \qquad \text{for all } v \in H^1_0(\Omega), \tag{3}$$

where $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ is the bilinear form given by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v, \qquad u, v \in H^1_0(\Omega), \tag{4}$$

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and $F : H^1_0(\Omega) \to \mathbb{R}$ is the linear functional defined by

$$F(v) = \int_\Omega fv, \qquad v \in H^1_0(\Omega). \tag{5}$$

Both a and F are clearly continuous, because

$$|a(u,v)| \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \leq \|u\|_{H^1} \|v\|_{H^1}, \tag{6}$$

and

$$|F(v)| \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2} \|v\|_{H^1}, \tag{7}$$

by the Cauchy-Bunyakowsky-Schwarz inequality. This means in particular that F is an element of the (topological) dual of $H^1_0(\Omega)$, which we denote by $H^{-1}(\Omega) \equiv [H^1_0(\Omega)]'$. In addition to continuity, the bilinear form a enjoys *strict coercivity* (also called H^1_0-ellipticity)

$$a(v,v) \geq c\|v\|^2_{H^1}, \qquad \text{for all } v \in H^1_0(\Omega), \tag{8}$$

with some constant $c > 0$, which is immediate from the Friedrichs inequality. Since a is symmetric, strict coercivity and continuity imply that a defines an inner product on $H^1_0(\Omega)$ which induces an equivalent norm. With these preparations at hand, we can now interpret the problem (3) in the following way: Given an element F from the dual space of $H^1_0(\Omega)$, can we represent F as the inner product $a(u, \cdot)$ with a fixed element $u \in H^1_0(\Omega)$? The affirmative answer to this question is precisely the content of the Riesz representation theorem.

Exercise 2 (Riesz representation theorem). Let H be a (real) Hilbert space, and let H' be its dual, defined as the space of continuous linear functionals on H. Let us denote the inner product of H by $\langle \cdot, \cdot \rangle$. Observe that any $y \in H$ defines an element $f \in H'$ by $f(x) = \langle y, x \rangle$ for $x \in H$. This defines a map $J : H \to H'$. The *Riesz representation theorem* (for Hilbert spaces)[1] states that J is invertible, that is, any continuous linear functional on H can be realized through the inner product with an element of H. We would like to prove this theorem by using a variational method. Let $f \in H'$, and let

$$E(x) = \langle x, x \rangle - 2f(x), \qquad x \in H, \tag{9}$$

and consider the problem of finding a minimizer of E over H.

- Show that a minimizing sequence for E exists and is Cauchy in H.
- Demonstrate that the limit minimizes E over H.
- Denoting by $y \in H$ the minimizer, show that $\langle y, x \rangle = f(x)$ for all $x \in H$.

The following is immediate.

Theorem 3. *In the above setting, there exists a unique $u \in H^1_0(\Omega)$ satisfying*

$$a(u,v) = F(v) \qquad \textit{for all } v \in H^1_0(\Omega). \tag{10}$$

In other words, the Poisson problem (1) has a unique weak solution.

We can also solve the same problem with an inhomogeneous Dirichlet condition without much difficulty. We formulate the problem as follows. Let $g \in H^1(\Omega)$ be given, and find $u \in \{g\} + H^1_0(\Omega)$ satisfying

$$a(u,v) = F(v) \qquad \text{for all } v \in H^1_0(\Omega). \tag{11}$$

Recall that the (affine) set $\{g\} + H^1_0(\Omega)$ is by definition $\{g + v : v \in H^1_0(\Omega)\}$.

Corollary 4. *For any $g \in H^1(\Omega)$, the preceding problem has a unique solution.*

[1] There is another result called Riesz representation theorem that is about representing linear functionals on a space of continuous functions as measures.

Proof. We will reformulate the problem as a problem with homogeneous Dirichlet condition, therefore reducing it to the known case. Let $w \in H_0^1(\Omega)$ be the unique function satisfying

$$a(w, v) = F(v) - a(g, v) \qquad \text{for all } v \in H_0^1(\Omega). \tag{12}$$

Such a function exists because the map $v \mapsto F(v) - a(g, v)$ is in $[H_0^1(\Omega)]'$ as can be seen from

$$|F(v) - a(g, v)| \leq \|f\|_{L^2}\|v\|_{H^1} + \|g\|_{H^1}\|v\|_{H^1}. \tag{13}$$

It is obvious that $u = w + g \in \{g\} + H_0^1(\Omega)$ and $a(u, v) = F(v)$ for all $v \in H_0^1(\Omega)$. The uniqueness part is left as an exercise. □

Exercise 5. Give a weak formulation of the problem

$$\begin{cases} -\Delta u + tu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{14}$$

and then show that a unique weak solution exists if $t > c$, where $c < 0$ is a constant.

2. Essential and natural boundary conditions

Let $f \in L^2(\Omega)$ and $t \in \mathbb{R}$, and consider the problem of finding $u \in H^1(\Omega)$ such that

$$\int_\Omega (\nabla u \cdot \nabla v + tuv) = \int_\Omega fv \qquad \text{for all } v \in H^1(\Omega). \tag{15}$$

What we did is that we took the problem (2), added the term tuv inside the integral, and replaced the space $H_0^1(\Omega)$ by $H^1(\Omega)$. The bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$a(u, v) = \int_\Omega (\nabla u \cdot \nabla v + tuv), \tag{16}$$

is symmetric and continuous, and linear functional $F : H^1(\Omega) \to \mathbb{R}$ defined by

$$F(v) = \int_\Omega fv, \tag{17}$$

is continuous, meaning that $F \in [H^1(\Omega)]'$. Moreover, it is easy to see that if $t > 0$ then

$$a(u, u) \geq \min\{1, t\}\|u\|_{H^1}^2, \qquad u \in H^1(\Omega), \tag{18}$$

i.e., a is strictly coercive in $H^1(\Omega)$. Hence by the Riesz representation theorem, the problem (15) has a unique solution $u \in H^1(\Omega)$.

Let us try to identify the classical problem corresponding to (15). Since $\mathscr{D}(\Omega) \subset H^1(\Omega)$, if we assume $u \in C^2(\Omega)$, then taking arbitrary $v \in \mathscr{D}(\Omega)$ in (15) implies that

$$-\Delta u + tu = f \qquad \text{a.e. in } \Omega. \tag{19}$$

We see that replacing the space $H_0^1(\Omega)$ by $H^1(\Omega)$ in the variational formulation has no effect on the differential equation to be satisfied in the interior of the domain at the classical level. If anything has changed, it must have something to do with the boundary condition. To probe what is happening at the boundary, in addition to $u \in C^2(\Omega)$, suppose that Ω has a C^1 boundary, and that $u \in C^1(\bar{\Omega})$. Then taking into account that $C^1(\bar{\Omega}) \subset H^1(\Omega)$, for any $v \in C^1(\bar{\Omega})$, we have

$$0 = \int_\Omega (\nabla u \cdot \nabla v + tuv - fv) = \int_{\partial\Omega} v\partial_\nu u + \int_\Omega (-\Delta u + tu - f)v = \int_{\partial\Omega} v\partial_\nu u, \tag{20}$$

where we have used (19), and ∂_ν is the (outward) normal derivative at $\partial\Omega$. It is not difficult to conclude from here that $\partial_\nu u = 0$ on $\partial\Omega$. So we identify (15) as a weak (or variational) formulation of the equation (19) with the homogeneous *Neumann boundary condition*.

Exercise 6. Show that if $u \in C^1(\bar{\Omega})$ satisfies

$$\int_{\partial\Omega} v\partial_\nu u = 0, \tag{21}$$

for all $v \in C^1(\bar{\Omega})$ then $\partial_\nu u = 0$ on $\partial\Omega$.

Remark 7. Note that the boundary term that arises from integration by parts in (20) naturally has lead us to the Neumann boundary condition. In variational terminology, such boundary conditions are called *natural boundary conditions*. If we were dealing with the Dirichlet boundary condition as in the previous section, on the contrary, the boundary term in (20) would be 0 because of the restriction $v \in H_0^1(\Omega)$ as opposed to $v \in H^1(\Omega)$. In this case, the boundary term vanishes by design of the underlying Hilbert space $H_0^1(\Omega)$, and the associated boundary conditions are called *essential boundary conditions*. Finally, we remark that the correspondences natural – Neumann and essential – Dirichlet do not always hold. For instance, in some formulations the Dirichlet boundary condition arises as the natural boundary condition.

3. Weak, strong, and classical solutions

In all cases considered so far, we have functions $u \in H^1(\Omega)$ and $f \in L^2(\Omega)$ satisfying

$$\int_\Omega (\nabla u \cdot \nabla v + tuv) = \int_\Omega fv \qquad \text{for all } v \in \mathscr{D}(\Omega). \tag{22}$$

Note that if (22) holds, then it holds also for all $v \in H_0^1(\Omega)$ by density. The differential equation associated to this is $-\Delta u + tu = f$ in Ω. As for the boundary condition, it is determined by the following additional information:

- For the homogeneous Dirichlet condition, we have the requirement $u \in H_0^1(\Omega)$.
- For the homogeneous Neumann condition, (22) holds also for all $v \in H^1(\Omega) \setminus H_0^1(\Omega)$.

We have encountered at least two different notions of solutions: Weak and classical. Here we want to formalize those concepts and introduce one more notion of a solution.

Definition 8. A *classical solution* of $-\Delta u + tu = f$ in Ω is a function $u \in C^2(\Omega)$ that satisfies the same equation pointwise in Ω. In particular, this would imply that $f \in C(\Omega)$.

Bear in mind that in the preceding definition we are concerned with only the differential equation, that is (supposed to be) satisfied at each point of the interior of the domain. When we are *solving* the boundary value problem, i.e., at the stage where we want to establish the existence of a solution, it is necessary to consider both the differential equation and the boundary condition at the same time. However, once the existence is known, when we want to study the properties of solutions, it is often possible and convenient to separate the two. Here we would like to consider conditions that are generalizations of the differential equation alone. In doing so, it is preferable to avoid global conditions, such as $u \in H^1(\Omega)$, which requires in particular the square integrability of u over Ω. Therefore it would be ideal if we replace H^1 by its local version, the same way $C(\Omega)$ is the local version of the space of bounded continuous functions on Ω. To this end, we let $H^1_{\text{loc}}(\Omega) = \{u \in L^2(\Omega) : \phi u \in H^1(\Omega), \forall \phi \in \mathscr{D}(\Omega)\}$.

Definition 9. We call $u \in H^1_{\text{loc}}(\Omega)$ a *weak solution* of $-\Delta u + tu = f$ in Ω if u satisfies (22). This notion makes sense even for $f \in L^2_{\text{loc}}(\Omega)$.

It is always true that classical solutions are weak. Moreover, under the assumption that $u \in C^2(\Omega)$ and $f \in C(\Omega)$, u is a classical solution if and only if it is a weak solution.

Now we generalize the definition of Sobolev spaces to higher order cases. We let

$$H^k(\Omega) = \{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega),\ |\alpha| \leq k\}, \tag{23}$$

equipped with the norm

$$\|u\|_{H^k} = \Big(\sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^2}^2 \Big)^{\frac{1}{2}}, \tag{24}$$

and the seminorm

$$|u|_{H^k} = \Big(\sum_{|\alpha|= k} \|\partial^\alpha u\|_{L^2}^2 \Big)^{\frac{1}{2}}. \tag{25}$$

Furthermore, we define $H^k_{\mathrm{loc}}(\Omega) = \{u \in L^2(\Omega) : \phi u \in H^k(\Omega), \forall \phi \in \mathscr{D}(\Omega)\}$.

Definition 10. We call $u \in H^2_{\mathrm{loc}}(\Omega)$ a *strong L^2 solution* of $-\Delta u + tu = f$ in Ω if the same equation is satisfied as an equality in the space $L^2_{\mathrm{loc}}(\Omega)$.

It is obvious that classical solutions are strong, and strong solutions are weak. Moreover, under the assumption that $u \in C^2(\Omega)$ and $f \in C(\Omega)$, u is a classical solution if and only if it is a strong solution. Similarly, under the assumption that $u \in H^2_{\mathrm{loc}}(\Omega)$ and $f \in L^2_{\mathrm{loc}}(\Omega)$, u is a strong solution if and only if it is a weak solution.

The above discussion leads to the question if and when we can guarantee that a weak solution u, initially only known to be in $H^1_{\mathrm{loc}}(\Omega)$, is indeed in $H^2_{\mathrm{loc}}(\Omega)$ or even in $C^2(\Omega)$, so that the weak solutions we have constructed in the previous sections are in fact strong or even classical. To see that it is not a completely absurd hope, think of the ordinary differential equation $y'' = f$, which has the property that if $f \in C^k$ then $y \in C^{k+2}$. Now if we assume that the equation $\Delta u = f$ has a similar property, and that its weak formulation inherits enough of its structure, then it is reasonable to expect that $f \in H^k_{\mathrm{loc}}(\Omega)$ implies $u \in H^{k+2}_{\mathrm{loc}}(\Omega)$, and that $f \in C^k(\Omega)$ implies $u \in C^{k+2}(\Omega)$, for a weak solution $u \in H^1_{\mathrm{loc}}(\Omega)$. In what follows we will show that this expectation is realized for the H^k spaces. However, it turns out that the statement in the C^k spaces is *not* true. We interpret this fact as the C^k spaces not being well suited for studying the Laplace operator. What we can establish are conditions on f (that are slightly stronger than $f \in C^k$) that imply $u \in C^{k+2}$.

4. Finite differences

In this section, we will establish a convenient criterion for determining whether a function is in a Sobolev space, which will then be used in the upcoming sections to study the Sobolev regularity of weak solutions to (22). Let $\Omega \subset \mathbb{R}^n$ be an open set and let f be a function defined on Ω. Then we define the *finite difference operator* Δ_h for $h \in \mathbb{R}^n$ by

$$\Delta_h f(x) = f(x+h) - f(x), \tag{26}$$

where we assume that $[x, x+h] \subset \Omega$, with $[x, x+h]$ denoting the line segment connecting x and $x+h$. Hence Δ_h maps functions on Ω to functions on the subset $\Omega_h = \{x \in \Omega : [x, x+h] \subset \Omega\}$. It is clear that

- f is uniformly continuous in Ω if and only if $\|\Delta_h f\|_{L^\infty(\Omega_h)} \to 0$ as $h \to 0$.
- f is Lipschitz continuous if and only if $\|\Delta_h f\|_{L^\infty(\Omega_h)} \leq c|h|$ $\forall h$, with some constant c.
- If $\|\Delta_h f\|_{L^\infty(\Omega_h)} = o(h)$ as $h \to 0$ then f is constant.

Since $C(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$, and we have $\|\Delta_h f\|_{L^p(\Omega_h)} \leq 2\|f\|_{L^p(\Omega)}$ for $f \in C(\Omega) \cap L^p(\Omega)$, the difference operator Δ_h can be uniquely extended to a continuous operator $\Delta_h : L^p(\Omega) \to L^p(\Omega_h)$. Alternatively, and with the same result, we can define $\Delta_h f = f \circ \tau_h - f$ for $f \in L^p(\Omega)$, where τ_h is the translation operator given by $\tau_h(x) = x + h$.

It will turn out that the condition $\|\Delta_h f\|_{L^2(\Omega_h)} = O(h)$ characterizes the Sobolev space $H^1(\Omega)$. Hence in a certain sense, $H^1(\Omega)$ is the L^2-equivalent of the Lipschitz space $C^{0,1}(\Omega)$. For generality, we will prove this characterization in the L^p setting.

Definition 11. We define $W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega),\, |\alpha| \leq k\}$, for $1 \leq p \leq \infty$ and $k \in \mathbb{N}_0$.

Note that we have $H^k(\Omega) = W^{k,2}(\Omega)$ and $L^p(\Omega) = W^{0,p}(\Omega)$.

Lemma 12. *If $f \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$, then $\|\Delta_h f\|_{L^p(\Omega_h)} \leq |h| \|\nabla f\|_{L^p(\Omega)}$ for $h \in \mathbb{R}^n$.*

Proof. For $\eta \in S^{n-1}$ and $f \in C^1(\Omega)$, by the fundamental theorem of calculus, we have

$$|f(x + \eta t) - f(x)| \leq \int_0^t |\nabla f(x + \eta s)| \,\mathrm{d}s \leq t^{1-\frac{1}{p}} \left(\int_0^t |\nabla f(x + \eta s)|^p \,\mathrm{d}s \right)^{\frac{1}{p}}. \tag{27}$$

Taking the p-th power and integrating, we get

$$\|\Delta_{\eta t} f\|^p_{L^p} \leq t^{p-1} \int_{\Omega_{\eta t}} \int_0^t |\nabla f(x + \eta s)|^p \,\mathrm{d}s \,\mathrm{d}x \leq t^{p-1} \int_0^t \int_\Omega |\nabla f(y)|^p \,\mathrm{d}y \,\mathrm{d}s, \tag{28}$$

which immediately leads us to the claim. □

In the converse direction, for $p > 1$, the decay condition $\|\Delta_{e_i t} f\|_{L^p} = O(t)$ implies the existence of $\partial_i f$ and the membership $\partial_i f \in L^p(\Omega)$. Here e_i is the i-th standard unit vector.

Theorem 13. *Let $f \in L^p(\Omega)$ with $1 < p < \infty$, and assume that $\|\Delta_{e_i t} f\|_{L^p(\Omega_h)} \leq Mt$ for all small $t > 0$. Then the weak derivative $\partial_i f$ exists and it satisfies $\|\partial_i f\|_{L^p(\Omega)} \leq M$.*

Proof. Let $K \subset \Omega$ be a compact set, and let $\varphi \in \mathscr{D}_K$, where $\mathscr{D}_K = \{\phi \in \mathscr{D}(\Omega) : \operatorname{supp} \phi \subset K\}$. Then we have

$$\int_\Omega \Delta_{e_i t} f \varphi = \int_\Omega (f(x + e_i t)\varphi(x) - f(x)\varphi(x)) \,\mathrm{d}x = -\int_\Omega f \Delta_{e_i t} \varphi, \tag{29}$$

for sufficiently small $t > 0$, which implies

$$\begin{aligned} \Big| \int_\Omega f \partial_i \varphi \Big| &\leq \Big| \int_\Omega \frac{\Delta_{e_i t} f}{t} \varphi \Big| + \Big| \int_\Omega f \big(\frac{\Delta_{e_i t} \varphi}{t} - \partial_i \varphi \big) \Big| \\ &\leq t^{-1} \|\Delta_{e_i t} f\|_{L^p(K)} \|\varphi\|_{L^q(K)} + \|f\|_{L^1} \Big\| \frac{\Delta_{e_i t} \varphi}{t} - \partial_i \varphi \Big\|_{L^\infty}, \end{aligned} \tag{30}$$

where q is the Hölder conjugate of p, i.e., it is given by $\frac{1}{p} + \frac{1}{q} = 1$. Now by using the bound $\|\Delta_{e_i t} f\|_{L^p} \leq Mt$ and sending $t \to 0$, we obtain

$$\Big| \int_\Omega f \partial_i \varphi \Big| \leq M \|\varphi\|_{L^q(K)}. \tag{31}$$

It shows that if we define the linear map $T : \mathscr{D}_K \to \mathbb{R}$ by

$$T\varphi = \int_\Omega f \partial_i \varphi, \tag{32}$$

then it is bounded in the L^q-norm: $|T\varphi| \leq M \|\varphi\|_{L^q(K)}$. Since $\mathscr{D}_K$ is dense in $L^q(K)$, the map T extends[2] uniquely to a continuous map $T : L^q(K) \to \mathbb{R}$. Moreover, the resulting map is linear, and satisfies the same bound $|T\varphi| \leq M \|\varphi\|_{L^q(K)}$ for $\varphi \in L^q(K)$. In other words, T is in the topological dual of $L^q(K)$, and from the duality between L^p and L^q, we conclude that there exists $g \in L^p(K)$ with $\|g\|_{L^p} \leq M$ such that

$$\int_K g\varphi = T\varphi = \int_\Omega f \partial_i \varphi, \qquad \text{for all } \varphi \in \mathscr{D}_K. \tag{33}$$

Hence, $-g$ is the weak derivative of f in the interior of K. In order to turn it into a global result, let $K_1 \subset K_2 \subset \ldots \subset \Omega$ be a sequence of compact sets such that $\bigcup_m K_m = \Omega$. Let $g_m = \partial_i f$ in the interior of K_m, and suppose that we extended g_m by 0 outside K_m. Moreover,

[2] Alternatively, we can use the Hahn-Banach theorem to extend T.

let us work with an arbitrary but fixed representative of $g_m \in L^p(\Omega)$, for each m. Then by uniqueness of the weak derivative it is clear that g_m agrees almost everywhere in K_m with all g_k with $k > m$, and hence the pointwise limit $g = \lim g_m$ exists almost everywhere in Ω, which of course represents the weak derivative $\partial_i f$ in Ω. Finally, since $\|g_m\|_{L^p} \leq M$ for all m, the monotone convergence theorem guarantees that $\|g\|_{L^p} \leq M$. □

Remark 14. This proof would not work for $p = 1$, because $L^1(\Omega)$ is not a dual space.

Exercise 15. Show that if $\|\Delta_h f\|_{L^p} = o(h)$ then f is a constant function.

Lemma 12 and Theorem 13 give the following characterization of $W^{1,p}$.

Corollary 16. *For $1 < p < \infty$, we have $f \in W^{1,p}(\Omega)$ if and only if $\|\Delta_h f\|_{L^p} = O(h)$.*

Remark 17. If we define the generalized Lipschitz spaces

$$\mathrm{Lip}(\alpha, L^p(\Omega)) = \{f \in L^p(\Omega) : \sup_{h \neq 0} |h|^{-\alpha} \|\Delta_h f\|_{L^p} < \infty\}, \tag{34}$$

then the preceding theorem says that in fact $W^{1,p}(\Omega) = \mathrm{Lip}(1, L^p(\Omega))$ for $1 < p < \infty$. Hence informally speaking, the Sobolev spaces are L^p-versions of Lipschitz spaces. The spaces $\mathrm{Lip}(\alpha, L^p(\Omega))$ with $0 < \alpha < 1$ are examples of *Nikolsky spaces*, which in some sense fill up the gap between L^p and $W^{1,p}$. Nikolsky spaces themselves are special cases of *Besov spaces*.

5. Interior H^k regularity

Now we start tackling the regularity problem for the equation (22) head on. To illustrate the main ideas clearly, we begin with a very simple case. Let $\Omega \subset \mathbb{R}^n$ be an open set, and recall that the *support* of a function $u \in L^1_{\mathrm{loc}}(\Omega)$ is defined as

$$\operatorname{supp} u = \Omega \setminus \bigcup \{\omega \subset \Omega \text{ open} : u|_\omega = 0\}. \tag{35}$$

In particular, $\operatorname{supp} u$ is a relatively closed set in Ω, and $u = 0$ almost everywhere in $\Omega \setminus \operatorname{supp} u$. For a compact set $K \subset \Omega$, let us introduce the space $H^1_K = \{u \in H^1(\Omega) : \operatorname{supp} u \subset K\}$. We have $H^1_K \subset H^1_0(\Omega)$ because for $u \in H^1_K$ and for all small $\varepsilon > 0$, the mollified function u_ε will have a compact support in Ω. Suppose that $u \in H^1_K$ and $f \in L^2(\Omega)$ satisfy

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \qquad \text{for all } v \in H^1_0(\Omega). \tag{36}$$

Recall that with τ_h defined by $\tau_h(x) = x + h$, we have $\Delta_h u = \tau_h^* u - u$, where $\tau_h^* u = u \circ \tau_h$ is the pull-back of u by τ_h. With $K' \subset \Omega$ a compact set containing K in its interior, since $\tau_h^* v \in H^1_0(\Omega)$ for $v \in H_{K'}$ and for all small $h \in \mathbb{R}^n$, it is easy to see that

$$\int_\Omega \nabla \tau_h^* u \cdot \nabla v = \int_\Omega \nabla u \cdot \nabla \tau_{-h}^* v = \int_\Omega f \tau_{-h}^* v \qquad \text{for all } v \in H_{K'}. \tag{37}$$

Combining this with (36) gives

$$\int_\Omega \nabla \Delta_h u \cdot \nabla v = \int_\Omega f \Delta_{-h} v \qquad \text{for all } v \in H_{K'}. \tag{38}$$

Then upon applying Lemma 12, we have

$$\Big| \int_\Omega \nabla \Delta_h u \cdot \nabla v \Big| \leq \|f\|_{L^2(\Omega)} \|\Delta_{-h} v\|_{L^2(\Omega)} \leq |h| \|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}, \tag{39}$$

for all small h and for all $v \in H_{K'}$. Now we put $v = \Delta_h u$, which is justified because $\Delta_h u \in H^1_{K'}$ for small h, and get

$$\|\nabla \Delta_h u\|_{L^2(\Omega)} \leq |h| \|f\|_{L^2(\Omega)}. \tag{40}$$

This in particular implies that

$$\|\Delta_{e_i t}\partial_j u\|_{L^2(\Omega)} \leq t\|f\|_{L^2(\Omega)}, \tag{41}$$

for all small $t > 0$, which means by Theorem 13 that the weak derivative $\partial_i\partial_j u$ exists and

$$\|\partial_i\partial_j u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}, \tag{42}$$

for all i, j. To conclude, if $u \in H^1_K$ and $f \in L^2(\Omega)$ satisfy (36), then we have $u \in H^2(\Omega)$ with $|u|_{H^2(\Omega)} \leq n\|f\|_{L^2(\Omega)}$. This is not fully satisfactory as it involves the unreasonable assumption $u \in H^1_K$, but the result can easily be generalized as follows.

Lemma 18. *Let $u \in H^1_{\text{loc}}(\Omega)$ and $f \in L^2_{\text{loc}}(\Omega)$ satisfy*

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \qquad \text{for all } v \in \mathscr{D}(\Omega), \tag{43}$$

and let $\eta \in \mathscr{D}(\Omega)$. Then we have

$$\int_\Omega \nabla(\eta u) \cdot \nabla v = \int_\Omega (\eta f - 2\nabla\eta \cdot \nabla u - u\Delta\eta)v \qquad \text{for all } v \in H^1_0(\Omega). \tag{44}$$

Hence the result we have established just before this lemma shows that $\eta u \in H^2(\Omega)$, and

$$|\nabla(\eta u)|_{H^2} \leq n\|\eta f\|_{L^2} + 2n\|\nabla\eta \cdot \nabla u\|_{L^2} + n\|u\Delta\eta\|_{L^2}. \tag{45}$$

In particular, we have $u \in H^2_{\text{loc}}(\Omega)$.

Proof. We have $\nabla(u\eta) = u\nabla\eta + \eta\nabla u$ in the weak sense because

$$\int_\Omega \eta u \partial_i\varphi = \int_\Omega u\partial_i(\eta\varphi) - \int_\Omega u\varphi\partial_i\eta, \qquad \varphi \in \mathscr{D}(\Omega). \tag{46}$$

Thus for $v \in \mathscr{D}(\Omega)$, we can write

$$\begin{aligned}\int_\Omega \nabla(\eta u) \cdot \nabla v = \int_\Omega (u\nabla\eta + \eta\nabla u) \cdot \nabla v &= \int_\Omega u\nabla\eta \cdot \nabla v + \nabla u \cdot (\nabla(\eta v) - v\nabla\eta) \\ &= \int_\Omega u\nabla\eta \cdot \nabla v + f\eta v - v\nabla u \cdot \nabla\eta,\end{aligned} \tag{47}$$

where we have used the formula $\eta\nabla v = \nabla(\eta v) - v\nabla\eta$ in the second equality, and (43) in the last line. The reasoning (46) with η replaced by $\partial_i\eta$ leads to $\partial_i(u\partial_i\eta) = u\partial_i^2\eta + \partial_i\eta\partial_i u$ in the weak sense, which then implies that

$$\int_\Omega u\nabla\eta \cdot \nabla v = -\int_\Omega (u\Delta\eta + \nabla\eta \cdot \nabla u)v. \tag{48}$$

Plugging this into (47) establishes (44) for $v \in \mathscr{D}(\Omega)$. Since the both sides of (44) are continuous in the $H^1(\Omega)$ norm as functions of v, it is clear that (44) holds for all $v \in H^1_0(\Omega)$. $\square$

Now we establish a higher regularity result. It shows that at least in the H^k scale, the solution is as regular as it is allowed by the right hand side f.

Theorem 19. *Let $u \in H^1_{\text{loc}}(\Omega)$ and $f \in H^k_{\text{loc}}(\Omega)$ satisfy*

$$\int_\Omega (\nabla u \cdot \nabla v + tuv) = \int_\Omega fv \qquad \text{for all } v \in \mathscr{D}(\Omega), \tag{49}$$

where $t \in \mathbb{R}$ is a constant. Then $u \in H^{k+2}_{\text{loc}}(\Omega)$, and we have the recursive estimate

$$|u|_{H^{k+2}(B)} \leq |f|_{H^k(B')} + c\,r^{-1}|u|_{H^{k+1}(B')} + c\,r^{-2}|u|_{H^k(B')}, \tag{50}$$

where $B = B_R(x)$ and $B' = B_{R+r}(x)$ with $R > 0$ and $\overline{B'} \subset \Omega$, and c is a constant that depends only on n and t.

Proof. Testing in (49) with $\partial^\alpha v$ instead of v, we write

$$\int_\Omega \nabla u \cdot \nabla \partial^\alpha v = \int_\Omega (f - tu)\partial^\alpha v \qquad \text{for all } v \in \mathscr{D}(\Omega). \tag{51}$$

If we temporarily assume that $u \in H^{j+1}_{\mathrm{loc}}(\Omega)$ for some $j \leq k$, then for all multi-indices α with $|\alpha| \leq j$, we have $\partial^\alpha u \in H^1_{\mathrm{loc}}(\Omega)$, and

$$\int_\Omega f \partial^\alpha v = (-1)^{|\alpha|} \int_\Omega (\partial^\alpha f) v, \qquad \text{and} \qquad \int_\Omega \nabla u \cdot \nabla \partial^\alpha v = (-1)^{|\alpha|} \int_\Omega \nabla \partial^\alpha u \cdot \nabla v, \tag{52}$$

hence

$$\int_\Omega \nabla \partial^\alpha u \cdot \nabla v = \int_\Omega (\partial^\alpha f - t\partial^\alpha u) v \qquad \text{for all } v \in \mathscr{D}(\Omega). \tag{53}$$

Now Lemma 18 implies that $\partial^\alpha u \in H^2_{\mathrm{loc}}(\Omega)$, that is, $u \in H^{j+2}_{\mathrm{loc}}(\Omega)$, and repeated applications of this argument with $j = 0, \ldots, k$ implies $u \in H^{k+2}_{\mathrm{loc}}(\Omega)$.

For the estimate (50), consider a function $\eta \in \mathscr{D}(B')$ such that $0 \leq \eta \leq 1$ everywhere, $\eta \equiv 1$ in B, and $r\|\nabla \eta\|_{L^\infty} + r^2 \|\Delta \eta\|_{L^\infty} \leq C$ for some constant C that does not depend on any of R and r. Then an application of (45) finishes the proof. □

6. Sobolev's lemma

In the previous section, we have established an H^k regularity result, i.e., that $f \in H^k_{\mathrm{loc}}$ implies $u \in H^{k+2}_{\mathrm{loc}}$, for weak solutions of $-\Delta u + tu = f$. In this section, we would like to investigate what this result implies in terms of the C^k regularity of u, i.e., if and when we have embeddings of the sort $H^k_{\mathrm{loc}}(\Omega) \subset C^m(\Omega)$. We first look at the case $m = 0$ and a special class of domains, called *cones*.

Lemma 20. *Let ω be an open subset of the unit sphere S^{n-1}, and let*

$$Q = Q(\omega, h) = \{\xi t : \xi \in \omega,\, 0 < t < h\} \subset \mathbb{R}^n, \tag{54}$$

with some $h > 0$. Suppose that $k > \frac{n}{2}$. Then we have

$$|u(0)| \leq c\|u\|_{H^k(Q)}, \qquad \text{for all } u \in C^\infty(\bar{Q}), \tag{55}$$

where $c = c(k, n, |\omega|, h)$.

Proof. First, let us derive the *Cauchy formula for repeated integration* in one dimension. Suppose that $f \in C^\infty(\mathbb{R})$ is a function satisfying $f(h) = f'(h) = \ldots = f^{(k-1)}(h) = 0$. Then by repeated integration by parts, we have

$$\begin{aligned} f(0) &= -\int_0^h f'(t)\,\mathrm{d}t = \int_0^h f''(t)t\,\mathrm{d}t = -\frac{1}{2}\int_0^h f'''(t)t^2\,\mathrm{d}t = \ldots \\ &= \frac{(-1)^k}{(k-1)!}\int_0^h f^{(k)}(t)t^{k-1}\,\mathrm{d}t, \end{aligned} \tag{56}$$

which is the claimed formula.

Now we pick $\phi \in \mathscr{D}(B_h)$ with $\phi(0) = 1$, and set $v = \phi u$. Moreover, fix $\xi \in \omega$ and put $f(t) = v(\xi t)$ for $t \in [0, h]$. We estimate $f^{(k)}(t)$ as

$$|f^{(k)}(t)| = \big|[(\xi_1\partial_1 + \ldots + \xi_n\partial_n)^k v](\xi t)\big| \leq n^k \max_{|\alpha|=k} |\partial^\alpha v(\xi t)|, \tag{57}$$

and by the Cauchy formula, we get

$$|u(0)| = |f(0)| \leq \frac{1}{(k-1)!}\int_0^h |f^{(k)}(t)|t^{k-1}\,\mathrm{d}t \leq \int_0^h g_k(\xi t)t^{k-1}\,\mathrm{d}t, \tag{58}$$

where

$$g_k(x) = \frac{n^k}{(k-1)!} \max_{|\alpha|=k} |\partial^\alpha v(x)|, \qquad x \in Q. \tag{59}$$

Then integrating over $\xi \in \omega$ and changing the integration variable from $(\xi, t) \in \omega \times (0, h)$ to $x = \xi t \in Q$, we infer

$$\begin{aligned} |u(0)| &\leq \frac{1}{|\omega|} \int_\omega \int_0^h g_k(\xi t) t^{k-1} \,\mathrm{d}t \,\mathrm{d}^{n-1}\xi \leq \frac{1}{|\omega|} \int_Q g_k(x)|x|^{k-n} \,\mathrm{d}x \\ &\leq \frac{1}{|\omega|} \|g_k\|_{L^2(Q)} \left(\int_Q |x|^{2(k-n)} \,\mathrm{d}x \right)^{\frac{1}{2}}, \end{aligned} \tag{60}$$

where the last integral is finite since $2k > n$. Namely, we have

$$|u(0)| \leq c|v|_{H^k(Q)} = c|\phi u|_{H^k(Q)}, \tag{61}$$

with $c = c(k, n, |\omega|, h)$, concluding the proof. □

Now arbitrary m and a quite general class of domains can be treated with ease.

Definition 21. A domain $\Omega \subset \mathbb{R}^n$ is said to have the *cone property* if there exists a cone $Q = Q(\omega, h)$ as in the preceding lemma, such that each $x \in \Omega$ is a vertex of a cone $Q_x \subset \Omega$ congruent to Q (in other words, one can place Q in Ω by moving its vertex to x and by suitably rotating around x).

Theorem 22. *Let $\Omega \subset \mathbb{R}^n$ be a domain having the cone property, and let $k > \frac{n}{2} + m$. Then we have*

$$\|u\|_{C^m(\Omega)} \leq c\|u\|_{H^k(\Omega)} \qquad \text{for } u \in C^m(\Omega) \cap H^k(\Omega), \tag{62}$$

where $c = c(k - m, n, |\omega|, h)$. In particular, $H^k(\Omega) \hookrightarrow C^m_b(\Omega)$, the latter is the space of functions whose derivatives of order up to m are continuous and bounded in Ω.

Proof. Let $u \in C^m(\Omega) \cap H^k(\Omega)$. Then for $x \in \Omega$ and for α satisfying $|\alpha| \leq m$, we have

$$|\partial^\alpha u(x)| \leq c\|\partial^\alpha u\|_{H^{k-m}(Q_x)} \leq c\|\partial^\alpha u\|_{H^{k-m}(\Omega)} \leq c\|u\|_{H^k(\Omega)}, \tag{63}$$

with $c = c(k - m, n, |\omega|, h)$, which implies (62).

Now suppose that $u \in H^k(\Omega)$, and let $\{u_j\} \subset C^\infty(\Omega) \cap H^k(\Omega)$ be a sequence satisfying $u_j \to u$ in $H^k(\Omega)$ as $j \to \infty$. Then an application of (62) to the difference $u_j - u_m$ implies that $\{u_j\}$ is a Cauchy sequence in $C^m_b(\Omega)$. Hence by the completeness of $C^m_b(\Omega)$ there is $w \in C^m_b(\Omega)$ such that $u_j \to w$ in $C^m_b(\Omega)$. In particular, we have $u = w$ almost everywhere. □

Corollary 23. *If $f \in H^{k-2}_{\text{loc}}(\Omega)$ with $k > \frac{n}{2} + m$ and $k \geq 2$ then any weak solution $u \in H^1_{\text{loc}}(\Omega)$ of $-\Delta u + tu = f$ satisfies $u \in C^m(\Omega)$. In particular, $f \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$.*

Note that an element of $C^m_b(\Omega)$ does not necessarily have a continuous extension to $\bar{\Omega}$. As such a property is important in classically satisfying a boundary condition, before closing the section, we present a sufficient condition for the embedding $H^k(\Omega) \subset C^m(\bar{\Omega})$ to hold. Here $C^m(\bar{\Omega})$ is the space of $C^m(\Omega)$ functions whose derivatives of order up to m have continuous extensions to $\bar{\Omega}$.

Definition 24. A domain $\Omega \subset \mathbb{R}^n$ is said to have the *strong cone property* if there exist a cone $Q = Q(\omega, h)$ and constants $\lambda > 0$ and $\mu \geq 1$, such that each pair $y, z \in \Omega$ with $|y - z| < \frac{h}{2\mu}$ is the vertices of the cones $Q_y \subset \Omega$ and $Q_z \subset \Omega$ congruent to Q, and

$$|Q_y \cap Q_z \cap B_r(y) \cap B_r(z)| \geq \lambda |y - z|^n, \tag{64}$$

where $r = \mu|y - z|$.

Theorem 25. *Let $\Omega \subset \mathbb{R}^n$ be a domain having the strong cone property, and let $k > \frac{n}{2} + m$. Then we have $H^k(\Omega) \hookrightarrow C^m(\bar{\Omega})$.*

Proof. We will prove the case $m = 0$ only. Without loss of generality, assume that $y = 0$, and that $Q = Q_y$. Let $Q' = Q \cap B_r(y)$, and $Q'_z = Q_z \cap B_r(z)$. Then for u smooth and $x \in Q' \cap Q'_z$, we have

$$|u(0) - u(z)| \leq |u(0) - u(x)| + |u(x) - u(z)|, \tag{65}$$

and integrating this over $Q' \cap Q'_z$, we get

$$\begin{aligned} |Q' \cap Q'_z||u(0) - u(z)| &\leq \int_{Q' \cap Q'_z} |u(0) - u(x)| \,\mathrm{d}x + \int_{Q' \cap Q'_z} |u(x) - u(z)| \,\mathrm{d}x \\ &\leq \int_{Q'} |u(0) - u(x)| \,\mathrm{d}x + \int_{Q'_z} |u(x) - u(z)| \,\mathrm{d}x. \end{aligned} \tag{66}$$

We pick $\phi \in \mathscr{D}(B_h)$ with $\phi \equiv 1$ in $B_{h/2}$, and set $v = \phi u$. Moreover, put $f(t) = v(\xi t)$ for $\xi \in \omega$ and $t \in [0, h]$. Then the first integral on the right side of (66) may be estimated as

$$\int_{Q'} |u(0) - u(x)| \,\mathrm{d}x \leq \int_\omega \int_0^r \int_0^\rho |f'(t)| \rho^{n-1} \,\mathrm{d}t \,\mathrm{d}\rho \,\mathrm{d}^{n-1}\xi \leq \frac{r^n}{n} \int_\omega \int_0^r |f'(t)| \,\mathrm{d}t \,\mathrm{d}^{n-1}\xi. \tag{67}$$

Now from the Cauchy formula

$$|f'(t)| \leq \frac{1}{(k-2)!} \int_t^h |f^{(k)}(s)| (s-t)^{k-1} \,\mathrm{d}t \leq \frac{1}{(k-2)!} \int_t^h |f^{(k)}(s)| s^{k-1} \,\mathrm{d}t, \tag{68}$$

we infer

$$\int_0^r |f'(t)| \,\mathrm{d}t \leq \frac{1}{(k-2)!} \int_0^r |f^{(k)}(t)| t^{k-1} \,\mathrm{d}t + \frac{r}{(k-2)!} \int_r^h |f^{(k)}(t)| t^{k-2} \,\mathrm{d}t, \tag{69}$$

and plugging this into (67) we get

$$\int_{Q'} |u(0) - u(x)| \,\mathrm{d}x \leq c r^n \int_{Q'} g_k(x) |x|^{k-n} \,\mathrm{d}x + c r^{n+1} \int_{Q \setminus Q'} g_k(x) |x|^{k-n-1} \,\mathrm{d}x, \tag{70}$$

where $c = (k, n)$ is a constant and $g_k(x) = \max_{|\alpha| = k} |\partial^\alpha v(x)|$ for $x \in Q$. For the first term on the right hand side, we have

$$\int_{Q'} g_k(x) |x|^{k-n} \,\mathrm{d}x \leq \|g_k\|_{L^2(Q')} \left(\int_Q |x|^{2(k-n)} \,\mathrm{d}x \right)^{\frac{1}{2}} \leq c r^{k-n/2} \|g_k\|_{L^2(Q)} \tag{71}$$

provided that $k - \frac{n}{2} > 0$, where $c = c(k, n, |\omega|)$ is a constant. For the other term, we have

$$\int_{Q \setminus Q'} g_k(x) |x|^{k-n-1} \,\mathrm{d}x \leq \|g_k\|_{L^2(Q)} \left(\int_{Q \setminus Q'} |x|^{2(k-n-1)} \,\mathrm{d}x \right)^{\frac{1}{2}} \leq c r^{k-1-n/2} \|g_k\|_{L^2(Q)}, \tag{72}$$

if $k - \frac{n}{2} < 1$, and

$$\int_{Q \setminus Q'} g_k(x) |x|^{k-n-1} \,\mathrm{d}x \leq c \log \frac{h}{r} \|g_k\|_{L^2(Q)}, \tag{73}$$

if $k - \frac{n}{2} = 1$, where in both cases $c = c(k, n, |\omega|)$ is a constant.

The second integral on the right hand side of (66) can be estimated in the same way, and we conclude for $y, z \in \Omega$ with $|y - z| < \frac{h}{2\mu}$, that

$$|u(y) - u(z)| \leq c |y - z|^{\frac{1}{2}} \|u\|_{H^k(\Omega)}, \tag{74}$$

if n is odd and $k = [\frac{n}{2}] + 1$, and

$$|u(y) - u(z)| \leq c|y-z| \log \frac{h}{|y-z|} \|u\|_{H^k(\Omega)}, \tag{75}$$

if n is even and $k = \frac{n}{2} + 1$, with $c = c(n, |\omega|, \lambda)$. In any case, this implies that H^k functions are uniformly continuous, and hence can be extended continuously to the closure of Ω. □

Remark 26. The preceding proof shows that $H^k(\Omega) \hookrightarrow C^{k-[\frac{n}{2}]-1,\alpha}(\Omega)$, for $\alpha = \frac{1}{2}$ when n is odd, and for any $\alpha < 1$ when n is even.

7. Analyticity

We know from Corollary 23 that if $f \in C^\infty(\Omega)$ then any weak solution $u \in H^1_{\text{loc}}(\Omega)$ of $-\Delta u + tu = f$ satisfies $u \in C^\infty(\Omega)$. In this section, we want to investigate the question whether or not $f \in C^\omega(\Omega)$ would imply $u \in C^\omega(\Omega)$. As before, let $\Omega \subset \mathbb{R}^n$ be a domain. Recall that a function $f \in C^\infty(\Omega)$ is called *(real) analytic in* Ω, and written $f \in C^\omega(\Omega)$, if for any $y \in \Omega$ the Taylor series

$$f(x) = \sum_\alpha \frac{\partial^\alpha f(y)}{\alpha!} (x-y)^\alpha, \tag{76}$$

converges in an open set containing y. We need some preliminary results to characterize analyticity by local growth estimates on derivatives.

Lemma 27. *A function f is real analytic in Ω if and only if for any point $y \in \Omega$ there exist a ball $B = B_r(y)$ with $r > 0$ and $\bar{B} \subset \Omega$, and constants $\delta > 0$ and $M < \infty$ such that*

$$\|f\|_{C^m(B)} \leq M \frac{m!}{\delta^m} \qquad \textit{for all } m \in \mathbb{N}. \tag{77}$$

Proof. We prove first the "if" part of the lemma. Let $y \in \Omega$ be an arbitrary point, and assume that (77) is satisfied for some $B = B_r(y)$ as hypothesized in the statement of the lemma. Our goal is now by using the estimates (77) to show that the Taylor series (76) converges in a neighbourhood of y. Without loss of generality, let us assume that $y = 0$. Given $z \in B$, consider the function $g(t) = f(zt)$. Taylor's theorem tells us

$$f(z) = g(1) = \sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(m)}(s)}{m!}, \tag{78}$$

where $0 \leq s \leq 1$. Let us compute the derivatives of g. We have

$$\begin{aligned} g'(t) &= (z_1\partial_1 + \ldots + z_n\partial_n) f(zt), \\ g''(t) &= (z_1\partial_1 + \ldots + z_n\partial_n)^2 f(zt), \ldots \\ g^{(k)}(t) &= (z_1\partial_1 + \ldots + z_n\partial_n)^k f(zt) \\ &= \sum_{\alpha_1 \ldots + \alpha_n = k} \frac{k!}{\alpha_1! \ldots \alpha_n!} z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} f(zt) \\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} z^\alpha \partial^\alpha f(zt), \end{aligned} \tag{79}$$

by the multinomial theorem, so

$$f(z) = \sum_{|\alpha|<m} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha + \underbrace{\frac{(z_1\partial_1 + \ldots + z_n\partial_n)^m f(sz)}{m!}}_{R_m}, \tag{80}$$

with $0 \le s \le 1$. We can estimate the remainder term by

$$|R_m| \le M\delta^{-m} n^m |z|^m = M\big(\frac{n|z|}{\delta}\big)^m, \tag{81}$$

which tends to 0 if $|z| < \frac{\delta}{n}$. This proves the "if" part of the lemma.

For the other direction, we start by assuming that $f \in C^\omega(\Omega)$. Hence for any given $y \in \Omega$, the Taylor series (76) converges in a neighbourhood of y. In particular, this means that for some $\delta = \delta(y) > 0$ possibly depending on y, the mentioned series converges for x given by $x = y + (\delta, \delta, \ldots, \delta)$, and since each term of a convergent series must be bounded, there exists a constant $M < \infty$ such that

$$|\partial^\alpha f(y)| \le M\delta^{-|\alpha|}\alpha! \qquad \text{for all } \alpha. \tag{82}$$

This is almost what we want, but we need to remove the possible dependence of δ on y. To address this issue, we consider the power series

$$f(z) = \sum_\alpha \frac{\partial^\alpha f(y)}{\alpha!}\,(z-y)^\alpha, \tag{83}$$

for $z \in \mathbb{C}^n$, which is just (76) extended to the complex domain. Now because of the estimate (82), the complex power series (83) converges (absolutely and uniformly) in an open set containing the closure of the polydisk $D_r(y) = D_r(y_1) \times \ldots \times D_r(y_n) \subset \mathbb{C}^n$ for some $r > 0$, where $D_r(y_k) \in \mathbb{C}$ is the disk of radius r centred at y_k. This means that for each $k \in \{1, \ldots, n\}$ and for each fixed

$$(\zeta_1, \ldots, \zeta_{k-1}, \zeta_{k+1}, \ldots, \zeta_n) \in \overline{D}_r(y_1) \times \ldots \times \overline{D}_r(y_{k-1}) \times \overline{D}_r(y_{k+1}) \times \ldots \times \overline{D}_r(y_n) \subset \mathbb{C}^{n-1}, \tag{84}$$

the function $\zeta_k \mapsto f(\zeta)$ is complex analytic in a neighbourhood containing $\overline{D}_r(y_k)$. Let $C_r(y_k)$ be the boundary of $D_r(y_k)$, positively oriented. Then Cauchy's formula gives

$$f(z) = \frac{1}{2\pi i}\int_{C_r(y_k)} \frac{f(z_1, \ldots, z_{k-1}, \zeta_k, z_{k+1}, \ldots, z_n)}{\zeta_k - z_k}\,\mathrm{d}\zeta_k, \tag{85}$$

for any $z \in \overline{D}_r(y_1) \times \ldots \times \overline{D}_r(y_{k-1}) \times D_r(y_k) \times \overline{D}_r(y_{k+1}) \times \ldots \times \overline{D}_r(y_n)$. Hence for any $z \in D_r(y)$, we have

$$f(z) = \frac{1}{2\pi i}\int_{C_r(y_n)} \frac{f(z_1, \ldots, z_{n-1}, \zeta_n)}{\zeta_n - z_n}\,\mathrm{d}\zeta_n, \tag{86}$$

and by applying the formula (85) recursively, we derive

$$f(z) = \frac{1}{(2\pi i)^n}\int_{C_r(y_n)} \cdots \int_{C_r(y_1)} \frac{f(\zeta)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)}\,\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n. \tag{87}$$

Since z is in the interior of $D_r(y)$, and so the integrand is analytic in an open set containing $\partial D_r(y)$, we can differentiate under the integral sign and get

$$\partial_z^\alpha f(z) = \frac{1}{(2\pi i)^n}\int_{C_r(y_n)} \cdots \int_{C_r(y_1)} \frac{f(\zeta)}{(\zeta - z)^{\alpha+1}}\,\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n, \tag{88}$$

where $\alpha + 1 = (\alpha_1 + 1, \ldots, \alpha_n + 1)$. If $z_k \in D_{r/2}(y_k)$ and $\zeta_k \in C_r(y_k)$, then $|\zeta_k - z_k| \ge r/2$, so we have the estimate

$$|\partial_z^\alpha f(z)| \le 2^n \alpha! (2/r)^{|\alpha|} \|f\|_{L^\infty(D_r(y))}, \tag{89}$$

for $z \in D_{r/2}(y)$. Restricting z to $\mathbb{R}^n$, we get the desired estimate. □

The next step is to characterize analyticity by the growth of local Sobolev norms.

Lemma 28. *A function f is real analytic in Ω if and only if for any point $y \in \Omega$ there exist a ball $B = B_r(y)$ with $r > 0$ and $\bar{B} \subset \Omega$, and constants $\delta > 0$ and $M < \infty$ such that*

$$\|f\|_{H^k(B)} \leq M \frac{k!}{\delta^k} \qquad \text{for all } k \in \mathbb{N}. \tag{90}$$

Proof. If $f \in C^\omega(\Omega)$ then by the previous lemma we have for any point $y \in \Omega$ there exist a ball $B = B_r(y)$ with $r > 0$ and $\bar{B} \subset \Omega$, and constants $\delta > 0$ and $M < \infty$ such that

$$\|f\|_{C^m(B)} \leq M \frac{m!}{\delta^m} \qquad \text{for all } m \in \mathbb{N}. \tag{91}$$

Now by using the trivial estimate

$$\|f\|_{H^m(B)} \leq \|f\|_{C^m(B)} \cdot |B|^{\frac{1}{2}}, \tag{92}$$

we get the "only if" part of the lemma.

To prove the other direction, suppose that we have (90). The ball B clearly satisfies the cone property, and the parameters $|\omega|$ and h depend only on the size of the ball B. Hence for any $m \in \mathbb{N}$, Sobolev's lemma gives

$$\|f\|_{C^m(B)} \leq c\|f\|_{H^{m+p}(B)} \leq cM(m+p)!\delta^{-m-p}, \tag{93}$$

where $p = [\frac{n}{2}] + 1$ and c is a constant that depends only on p, n, and the size of the ball B. For $m \leq p$ we have

$$(m+p)! \leq (2p)! \leq 2^p p^{2p}, \tag{94}$$

and for $m > p$ we have

$$(m+p)! \leq m!(2m)^p \leq p^p e^m m!, \tag{95}$$

which imply that

$$\|f\|_{C^m(B)} \leq cM\delta^{-p} 2^p p^{2p} m!(\delta/e)^{-m}, \tag{96}$$

establishing the lemma. □

Finally, we present here the analytic regularity theorem.

Theorem 29. *Let $u \in H^1_{\text{loc}}(\Omega)$ be a weak solution of $-\Delta u + tu = f$, where $t \in \mathbb{R}$ and $f \in C^\omega(\Omega)$. Then $u \in C^\omega(\Omega)$.*

Proof. By Theorem 19, we have $u \in C^\infty(\Omega)$, so we only need to derive suitable estimates on the local Sobolev norms. Let $B = B_R(y)$ and $B' = B_{R+\rho}(y)$ with $R > 0$, $\rho > 0$ and $\overline{B'} \subset \Omega$. Let $K_\ell = B_{R+\ell r}(y)$ for $\ell = 0, \ldots, k$ with $r = \rho/k$. Note that $K_0 = B$ and $K_k = B'$. Then the estimate (50) from Theorem 19 yields

$$|u|_{H^{k-\ell}(K_\ell)} \leq |f|_{H^{k-\ell-2}(K_{\ell+1})} + (cr)^{-1}|u|_{H^{k-\ell-1}(K_{\ell+1})} + (cr)^{-2}|u|_{H^{k-\ell-2}(K_{\ell+1})}, \tag{97}$$

for $\ell = 0, \ldots, k-1$, where c is a constant that depends only on n and t. We infer

$$\begin{aligned} |u|_{H^k(B)} &\leq (cr)^{-k}\|u\|_{H^1(B')} + \sum_{\ell=0}^{k-2} (cr)^{\ell-k+2}|f|_{H^\ell(B')} \\ &\leq (cr)^{-k}\|u\|_{H^1(B')} + \sum_{\ell=0}^{k-2} (cr)^{\ell-k+2} M\frac{\ell!}{\delta^\ell}, \end{aligned} \tag{98}$$

by analyticity of f. Now by noting that $k^j \leq j!e^k$ for any $j \in \mathbb{N}$, we have

$$\begin{aligned} |u|_{H^k(B)} &\leq (c\rho)^{-k}e^k k!\|u\|_{H^1(B')} + \sum_{\ell=0}^{k-2} (c\rho)^{\ell-k+2} e^{k-\ell}(k-\ell)! M\frac{\ell!}{\delta^\ell} \\ &\leq (c\rho)^{-k}e^k k!\|u\|_{H^1(B')} + Ck!(\delta')^{-k}, \end{aligned} \tag{99}$$

for some constants C and $\delta' > 0$, which completes the proof. □

Exercise 30. (Gevrey regularity) We say that $f \in C^\infty(\Omega)$ is in the *Gevrey class* $G^\alpha(\Omega)$ with $\alpha \geq 1$, if for any ball B with $\bar{B} \subset \Omega$, there exist $\delta > 0$ and $M < \infty$ such that

$$\|f\|_{C^m(B)} \leq M \frac{(m!)^\alpha}{\delta^m} \qquad \text{for all } m \in \mathbb{N}. \tag{100}$$

We have $G^\alpha(\Omega) \subset G^\beta(\Omega)$ for $\alpha \leq \beta$, and $G^1(\Omega) = C^\omega(\Omega)$. Also, it makes sense to define $G^\infty = C^\infty$. Hence in some sense, the Gevrey classes fill the gap between C^ω and C^∞. Prove that if $f \in G^\alpha(\Omega)$ for some $\alpha \geq 1$ then any weak solution $u \in H^1_{\text{loc}}(\Omega)$ of $-\Delta u + tu = f$ satisfies $u \in G^\alpha(\Omega)$.

8. Regularity up to the boundary

So far we have been concerned with regularity results that are valid in compact subsets of the domain Ω. One of the most fundamental theorems in this direction is the fact that $f \in L^2_{\text{loc}}(\Omega)$ implies $u \in H^2_{\text{loc}}(\Omega)$. Such results are called *interior regularity* results, and they depend neither on the smoothness of the boundary nor on the boundary condition. In this section, we want to study the H^k regularity of u in sets of the form $\Omega \cap B$ where B is a ball centred at a point of $\partial\Omega$. A typical result we would like to have is the implication $f \in L^2(\Omega \cap B) \Rightarrow u \in H^2(\Omega \cap B)$, although it will turn out that one needs to impose some smoothness conditions on $\partial\Omega$.

We start with the half-space case. Note that the Dirichlet and Neumann boundary conditions are treated simultaneously.

Lemma 31. *Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, and let V be either $H^1_0(\mathbb{R}^n_+)$ or $H^1(\mathbb{R}^n_+)$. Suppose that $u \in V$ and $f \in L^2(\mathbb{R}^n_+)$ satisfy*

$$\int_{\mathbb{R}^n_+} \nabla u \cdot \nabla v = \int_{\mathbb{R}^n_+} fv \qquad \text{for all } v \in V. \tag{101}$$

Then we have $u \in H^2(\mathbb{R}^n_+)$ and $|u|_{H^2(\mathbb{R}^n_+)} \leq 2n\|f\|_{L^2(\mathbb{R}^n_+)}$.

Proof. Since $\tau^*_h v \in V$ for $v \in V$ and for all $h \in \mathbb{R}^n$ with $h_n = 0$, we have

$$\int_{\mathbb{R}^n_+} \nabla \tau^*_h u \cdot \nabla v = \int_{\mathbb{R}^n_+} \nabla u \cdot \nabla \tau^*_{-h} v = \int_{\mathbb{R}^n_+} f \tau^*_{-h} v \qquad \text{for all } v \in V, \tag{102}$$

and therefore

$$\int_{\mathbb{R}^n_+} \nabla \Delta_h u \cdot \nabla v = \int_{\mathbb{R}^n_+} f \Delta_{-h} v \qquad \text{for all } v \in V. \tag{103}$$

Then it follows from Lemma 12 that

$$\Big| \int_{\mathbb{R}^n_+} \nabla \Delta_h u \cdot \nabla v \Big| \leq \|f\|_{L^2(\mathbb{R}^n_+)} \|\Delta_{-h} v\|_{L^2(\mathbb{R}^n_+)} \leq |h| \|f\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)}, \tag{104}$$

for all h with $h_n = 0$ and for all $v \in V$. Now we put $v = \Delta_h u$, which is justified because $\Delta_h u \in V$ when $h_n = 0$, and get

$$\|\nabla \Delta_h u\|_{L^2(\mathbb{R}^n_+)} \leq |h| \|f\|_{L^2(\mathbb{R}^n_+)}. \tag{105}$$

This in particular implies that

$$\|\Delta_{e_i t} \partial_j u\|_{L^2(\mathbb{R}^n_+)} \leq t \|f\|_{L^2(\mathbb{R}^n_+)}, \tag{106}$$

for $t > 0$, $i = 1, \ldots, n-1$, and $j = 1, \ldots, n$, which means by Theorem 13 that the weak derivative $\partial_i \partial_j u$ exists and

$$\|\partial_i \partial_j u\|_{L^2(\mathbb{R}^n_+)} \leq \|f\|_{L^2(\mathbb{R}^n_+)}, \tag{107}$$

for $i = 1, \ldots, n-1$, and $j = 1, \ldots, n$.

In order to make the conclusion $u \in H^2(\mathbb{R}^n_+)$, the only missing ingredient at this point is to show that $\partial_n^2 u \in L^2(\mathbb{R}^n_+)$. Arguing formally, the equation $-\Delta u = f$ implies $-\partial_n^2 u = f + \sum_{i=1}^{n-1} \partial_i^2 u$, so if all $\partial_i^2 u$ for $i = 1, \ldots, n-1$ are in L^2, then $\partial_n^2 u$ ought to be in L^2. We can make sense of this formal argument as follows. The equation (101) implies

$$\int_{\mathbb{R}^n_+} \partial_n u \cdot \partial_n v = \int_{\mathbb{R}^n_+} f v - \sum_{i=1}^{n-1} \int_{\mathbb{R}^n_+} \partial_i u \cdot \partial_i v \qquad \text{for all } v \in \mathscr{D}(\mathbb{R}^n_+). \tag{108}$$

Then since $\partial_i u \in H^1(\mathbb{R}^n_+)$, by definition of the weak derivative we have

$$\int_{\mathbb{R}^n_+} \partial_i u \cdot \partial_i v = -\int_{\mathbb{R}^n_+} \partial_i^2 u \cdot v, \tag{109}$$

and therefore

$$\int_{\mathbb{R}^n_+} \partial_n u \cdot \partial_n v = \int_{\mathbb{R}^n_+} \big(f + \sum_{i=1}^{n-1} \partial_i^2 u\big) v \qquad \text{for all } v \in \mathscr{D}(\mathbb{R}^n_+). \tag{110}$$

By definition, this means that $\partial_n^2 u$ exists in the weak sense and equal to the expression in the brackets (up to a sign). Since the expression in the brackets is in $L^2(\mathbb{R}^n_+)$, we conclude that $\partial_n^2 u \in L^2(\mathbb{R}^n_+)$, and thus $u \in H^2(\mathbb{R}^n_+)$. □

Remark 32. We observe that the translation invariance of the space V (which is H^1_0 or H^1) along the boundary $\partial\mathbb{R}^n_+$ was important in making the finite difference argument work.

Next, we would like to extend the result to domains with curved boundaries. To this end, we consider a domain whose boundary is given by the graph of a function.

Theorem 33. *Let $\phi \in C^2_b(\mathbb{R}^{n-1})$, and let $\Omega = \{x \in \mathbb{R}^n : x_n > \phi(x_1, \ldots, x_{n-1})\}$. Let V be either $H^1_0(\Omega)$ or $H^1(\Omega)$. Suppose that $u \in V$ and $f \in L^2(\Omega)$ satisfy*

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \qquad \textit{for all } v \in V. \tag{111}$$

Then we have $u \in H^2(\Omega)$.

Proof. Under the coordinate transformation

$$\begin{cases} y_i = x_i, & i = 1, \ldots, n-1, \\ y_n = x_n - \phi(x_1, \ldots, x_{n-1}), \end{cases} \tag{112}$$

our domain becomes the upper half space $\mathbb{R}^n_+$. However, we cannot apply Lemma 31 directly, because the equation (111) might change under the transform (112). Indeed, the computation

$$\begin{cases} \partial_{x_i} u = \partial_{y_i} u - \partial_{y_n} u \cdot \partial_{x_i} \phi, & i = 1, \ldots, n-1, \\ \partial_{x_n} u = \partial_{y_n} u, \end{cases} \tag{113}$$

shows that the situation is not completely trivial. Let us write it as

$$\nabla_x u = B \nabla_y u \equiv \begin{pmatrix} I & -\nabla\phi \\ 0 & 1 \end{pmatrix} \nabla_y u, \tag{114}$$

where ∇_x and ∇_y designate gradients with respect to the x and y coordinates, respectively, and $\nabla\phi = (\partial_1 \phi, \ldots, \partial_{n-1}\phi)$. All gradients are understood as column vectors. Note that we have the explicit formula

$$B^{-1} = \begin{pmatrix} I & \nabla\phi \\ 0 & 1 \end{pmatrix}. \tag{115}$$

For smooth functions u and v, we have

$$\nabla_x u \cdot \nabla_x v = (\nabla_y u)^T A \nabla_x v \equiv \sum_{i,k} a_{ik} \partial_{y_i} u \cdot \partial_{y_k} v, \tag{116}$$

where a_{ik} $(i, k = 1, \ldots, n)$ are the elements of the matrix

$$A = B^T B = \begin{pmatrix} I & -\nabla\phi \\ -(\nabla\phi)^T & 1 \end{pmatrix}. \tag{117}$$

Since the Jacobian determinant of the transformation (112) is 1, we infer

$$\int_\Omega \nabla_x u \cdot \nabla_x v = \int_{\mathbb{R}^n_+} (\nabla_y u)^T A \nabla_y v. \tag{118}$$

This in particular implies that

$$\alpha \int_\Omega |\nabla_y u|^2 \leq \int_{\mathbb{R}^n_+} |\nabla_x u|^2 \leq \beta \int_\Omega |\nabla_y u|^2, \tag{119}$$

for some constants $\alpha > 0$ and $\beta < \infty$, because

$$\xi^T A \xi = |B\xi|^2 \leq \beta |\xi|^2 \qquad \text{for } \xi \in \mathbb{R}^n, \tag{120}$$

and

$$|\xi|^2 = |B^{-1} B \xi|^2 \leq \beta' |B\xi|^2 = \beta' \xi^T A \xi, \qquad \text{for } \xi \in \mathbb{R}^n, \tag{121}$$

where the constants β and β' depend only on $\|\nabla\phi\|_{L^\infty(\mathbb{R}^{n-1})}$. Hence, a function u is in $H^1(\Omega)$ with respect to the x coordinates if and only if u is in $H^1(\mathbb{R}^n_+)$ with respect to the y coordinates, and an analogous statement holds for the H^1_0 spaces. In other words, the pull-back map corresponding to the transformation (112) extends uniquely and continuously to an isomorphism between Sobolev spaces. We are going to reuse the letter V to also denote the same space with Ω replaced by $\mathbb{R}^n_+$. Then we have

$$\sum_{i,k} \int_{\mathbb{R}^n_+} a_{ik} \partial_i u \cdot \partial_k v = \int_{\mathbb{R}^n_+} f v \qquad \text{for all } v \in V. \tag{122}$$

From this point we proceed similarly to the proof of Lemma 31. Let $h \in \mathbb{R}^n$ be an arbitrary vector with $h_n = 0$. Since $\tau_h^* v \in V$ for $v \in V$, we have

$$\sum_{i,k} \int_{\mathbb{R}^n_+} \tau_h^*(a_{ik} \partial_i u) \partial_k v = \sum_{i,k} \int_{\mathbb{R}^n_+} a_{ik} \partial_i u \cdot \partial_k \tau_{-h}^* v = \int_{\mathbb{R}^n_+} f \tau_{-h}^* v \qquad \text{for all } v \in V, \tag{123}$$

and therefore

$$\sum_{i,k} \int_{\mathbb{R}^n_+} \Delta_h(a_{ik} \partial_i u) \partial_k v = \int_{\mathbb{R}^n_+} f \Delta_{-h} v \qquad \text{for all } v \in V. \tag{124}$$

Now taking into account that

$$\Delta_h(a_{ik} \partial_i u) = a_{ik} \partial_i \Delta_h u + \Delta_h a_{ik} \cdot \partial_i \tau_h^* u, \tag{125}$$

we infer

$$\sum_{i,k} \int_{\mathbb{R}^n_+} a_{ik} \partial_i \Delta_h u \cdot \partial_k v = \int_{\mathbb{R}^n_+} f \Delta_{-h} v - \sum_{i,k} \int_{\mathbb{R}^n_+} \Delta_h a_{ik} \cdot \partial_i \tau_h^* u \cdot \partial_k v \qquad \text{for all } v \in V. \tag{126}$$

Then Lemma 12 implies

$$\Big| \sum_{i,k} \int_{\mathbb{R}^n_+} a_{ik} \partial_i \Delta_h u \cdot \partial_k v \Big| \leq |h| \|f\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)} + M|h| \|\nabla u\|_{L^2(\mathbb{R}^n_+)} \|\nabla v\|_{L^2(\mathbb{R}^n_+)}, \tag{127}$$

for all $v \in V$, where M depends on the Lipschitz norm of a_{ik}. We put $v = \Delta_h u$, which is justified because $\Delta_h u \in V$, and get

$$\alpha \|\nabla \Delta_h u\|_{L^2(\mathbb{R}^n_+)} \leq |h| \|f\|_{L^2(\mathbb{R}^n_+)} + M|h| \|\nabla u\|_{L^2(\mathbb{R}^n_+)}, \tag{128}$$

with $\alpha > 0$ a constant. Since $h \in \mathbb{R}^n$ is an arbitrary vector with $h_n = 0$, we conclude by Theorem 13 that the weak derivative $\partial_i \partial_j u$ exists and

$$\alpha \|\partial_i \partial_j u\|_{L^2(\mathbb{R}^n_+)} \leq \|f\|_{L^2(\mathbb{R}^n_+)} + M \|\nabla u\|_{L^2(\mathbb{R}^n_+)}, \tag{129}$$

for $i = 1, \ldots, n-1$, and $j = 1, \ldots, n$.

Next, we shall show that $\partial_n^2 u \in L^2(\mathbb{R}^n_+)$. The equation (122) implies

$$\int_{\mathbb{R}^n_+} a_{nn} \partial_n u \cdot \partial_n v = \int_{\mathbb{R}^n_+} f v - \sum_{i+k<2n} \int_{\mathbb{R}^n_+} a_{ik} \partial_i u \cdot \partial_k v \qquad \text{for all } v \in \mathscr{D}(\mathbb{R}^n_+). \tag{130}$$

Recall that $a_{nn} = 1$. Since $\partial_i \partial_k u \in L^2(\mathbb{R}^n_+)$, we have

$$\int_{\mathbb{R}^n_+} a_{ik} \partial_i u \cdot \partial_k v = -\int_{\mathbb{R}^n_+} (\partial_k a_{ik} \cdot \partial_i u + a_{ik} \partial_i \partial_k u) v, \tag{131}$$

and therefore

$$\int_{\mathbb{R}^n_+} \partial_n u \cdot \partial_n v = \int_{\mathbb{R}^n_+} \big(f + \sum_{i+k<2n} (\partial_k a_{ik} \cdot \partial_i u + a_{ik} \partial_i \partial_k u)\big) v \qquad \text{for all } v \in \mathscr{D}(\mathbb{R}^n_+). \tag{132}$$

By definition, this means that $\partial_n^2 u$ exists in the weak sense and equal to the expression in the brackets (up to a sign). Since the expression in the brackets is in $L^2(\mathbb{R}^n_+)$, we conclude that $\partial_n^2 u \in L^2(\mathbb{R}^n_+)$, and thus $u \in H^2(\mathbb{R}^n_+)$. □

Exercise 34. Complete the proof by showing that $u \in H^2(\mathbb{R}^n_+)$ in the y-coordinates implies $u \in H^2(\Omega)$ in the x-coordinates.

Exercise 35. Relax the condition $\phi \in C^2_b(\mathbb{R}^{n-1})$ in the preceding theorem to $\phi \in C^{1,1}(\mathbb{R}^{n-1})$. Recall that the *Hölder (or Hölder-Lipschitz) space* $C^{k,\theta}(D)$ on a domain $D \subset \mathbb{R}^d$ is defined as the space of functions $\phi \in C^k(D)$ for which

$$\|\phi\|_{C^{k,\theta}(D)} = \sum_{|\beta| \leq k} \sup_{x \in D} |\partial^\beta \phi(x)| + \sum_{|\beta| = k} \sup_{x,y \in D} \frac{|\partial^\beta \phi(x) - \partial^\beta \phi(y)|}{|x-y|^\theta} < \infty.$$

9. The Newtonian potential

Roughly speaking, what we have learned from the regularity theory so far is that by solving the Poisson equation $\Delta u = f$, we gain exactly 2 orders of regularity in the H^k scale: $f \in H^k$ implies $u \in H^{k+2}$. A primitive form of such a phenomenon is the following fact for ordinary differential equations: If y solves $y'' = f$ and $f \in C^k$ then $y \in C^{k+2}$. Now if we want to obtain a regularity result for the Poisson equation in the C^k scale, as we already saw, one way is to use Sobolev's lemma to trade Sobolev regularity for classical regularity. Schematically, it would work as follows.

$$f \in C^k \quad \Rightarrow \quad f \in H^k \quad \Rightarrow \quad u \in H^{k+2} \quad \Rightarrow \quad u \in C^m. \tag{133}$$

However, this trade is not very efficient, as we need $k+2 > m + \frac{n}{2}$ in order to have $H^{k+2} \subset C^m$. For instance, if we want to guarantee $u \in C^2$, then we would need $f \in C^k$ with $k > \frac{n}{2}$. Already for $n = 2$ it means $k > 1$, and it gets worse as n increases. So the route through Sobolev's lemma in general does not allow any gain of regularity in the C^k scale, and in fact it even loses regularity in higher dimensions.

In this section, we will study the C^k regularity question by a direct approach that does not rely on the H^k regularity results. As we will see, the answer turns out to be a bit subtle. The subtlety is caused by the fact that in general we do not gain 2 orders of regularity in the C^k scale. It is illustrated by the following example which shows that there is a continuous function f for which the equation $\Delta u = f$ has no C^2 solution in any neighbourhood of 0.

Example 36 (Gilbarg-Trudinger)**.** Let $p(x) = x_1x_2$, so that we have $\Delta p = 0$ and $\partial_1\partial_2 p = 1$ in $\mathbb{R}^2$. In addition, let η be a smooth nonnegative function such that $\eta \equiv 1$ in B_1 and $\operatorname{supp}\eta \subset B_2$, and let $\{c_k\}$ be a sequence satisfying $\sum_k c_k = \infty$ and $c_k \to 0$ as $k \to \infty$. Define

$$u(x) = \sum_{k=0}^{\infty} c_k 2^{-2k} g(2^k x), \tag{134}$$

where $g = \eta p$. It is easy to see that $u \in C^1(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$, but u is *not* in $C^2(\mathbb{R}^2)$ because

$$\partial_1\partial_2 u(x) = \sum_{k=0}^{\infty} c_k (\partial_1\partial_2 g)(2^k x) = \sum_{k=0}^{\infty} c_k, \tag{135}$$

blows up at $x = 0$. Moreover, we have

$$f(x) := \Delta u(x) = \sum_{k=0}^{\infty} c_k (\Delta g)(2^k x) =: \sum_{k=0}^{\infty} f_k(x), \tag{136}$$

for $x \neq 0$. We see that the term f_k is supported in the annulus $\{2^{-k} \leq |x| < 2^{1-k}\}$, and $\|f_k\|_{L^\infty} \to 0$ as $k \to \infty$. Hence $f \in C(\mathbb{R}^n)$. Now assume that $v \in C^2(B_\varepsilon)$ satisfies $\Delta v = f$ in B_ε, for some $\varepsilon > 0$. Then $u - v \in C^2(B_\varepsilon \setminus \{0\})$ is harmonic in $B_\varepsilon \setminus \{0\}$. Obviously $u - v$ is bounded in a neighbourhood of 0, which means by the removable singularity theorem that $u = v$ in B_ε. However, $u \notin C^2$, and so no such v exists.

Exercise 37. (Yudovich) Show that the function $u \in C^\infty(\mathbb{D} \setminus \{0\})$ given by

$$u(x) = x_1x_1 \log\log|x|^{-2}, \tag{137}$$

satisfies $u \notin C^2(\mathbb{D})$ and $\Delta u \in C(\mathbb{D})$.

Exercise 38. Let $p(x) = x_1^3 - 3x_1x_2^2$ in $\mathbb{R}^2$, and let η and $\{c_k\}$ be as in Example 36. Let

$$f(x) = \sum_{k=0}^{\infty} c_k 2^{-k} (\Delta g)(2^k x), \tag{138}$$

where $g = \eta p$. Show that $g \in C^1$ but that $\Delta u = f$ does not have a $C^{2,1}$ solution in any neighbourhood of 0.

In retrospect, Example 36 is not so surprising because the condition $\Delta u = f \in C$ means only that a particular combination of the second derivatives of u is continuous, while $u \in C^2$ means that each second derivative of u is continuous. To see that solving a partial differential equation does not always gain regularity, consider the equation $\partial_x^2 u = f$ in $\mathbb{R}^2$. Clearly this equation does not gain any regularity in the y-direction, because, intuitively speaking, it does not mix things up in the y-direction. That being said, the equations $\Delta u = f$ and $\partial_x^2 u = f$ are completely different in nature (when $n \geq 2$). The equation $\Delta u = f$ does indeed mix things up in all directions, and this is the reason why we have good regularity properties in Sobolev spaces. To contrast, the equation $\partial_x^2 u = f$ in $\mathbb{R}^2$ does not gain regularity in the Sobolev scale (except the obvious smoothing only in the x-direcion). So one could argue that the real surprise is the Sobolev regularity properties of the Poisson equation. In fact, even in the C^k scale the situation is not so bad. We need to impose on f a condition that is only slightly stronger than $f \in C^k$ in order to get $u \in C^{k+2}$. Moreover, this slight flaw can be "rectified" by working in Hölder spaces: If $f \in C^{k,\alpha}$ with $0 < \alpha < 1$ then $u \in C^{k+2,\alpha}$.

Let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a weak solution of $\Delta u = f$ in Ω. Then we observe that if $v \in H^1_{\mathrm{loc}}(\Omega)$ is another weak solution of $\Delta v = f$ in Ω, which we explicitly construct or we know a great deal about, then $u - v$ is a weak solution of $\Delta(u - v) = 0$ in Ω, hence by Weyl's lemma $u - v \in C^\omega(\Omega)$ and $u - v$ is harmonic in the classical sense. So the regularity of u is as good as that of v. A good candidate for v is the *Newtonian potential* of f:

$$v(x) = \int_\Omega E(x - y) f(y)\,\mathrm{d}y, \tag{139}$$

where we recall that the fundamental solution E is given by

$$E(x) = \begin{cases} \frac{1}{(2-n)|S^{n-1}|}|x|^{2-n} & \text{if } n \neq 2, \\ \frac{1}{2\pi}\log|x| & \text{if } n = 2. \end{cases} \tag{140}$$

Remark 39. If we can show that the function v satisfies $v \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\Delta v = f \in C(\Omega)$, then it gives a way to solve the Poisson problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{141}$$

by Perron's method. The idea is to solve the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ w = g - v & \text{on } \partial\Omega, \end{cases} \tag{142}$$

first and then put $u = w + v$.

For convenience, if it is possible, we extend f continuously to a compactly supported function in $\mathbb{R}^n$. If it is not possible, we perform such an extension after restricting f to a ball whose closure is contained in Ω. Hence in the following, we are going to assume that $\Omega = \mathbb{R}^n$, and fix some $f \in C(\mathbb{R}^n)$ with compact support. Note that (139) becomes now

$$v(x) = \int_{\mathbb{R}^n} E(x - y) f(y)\,\mathrm{d}y = (E * f)(x). \tag{143}$$

We begin by making some simple observations.

Lemma 40. *We have $v \in C^1(\mathbb{R}^n)$ and $\Delta v = f$ in $\mathbb{R}^n$ in the weak sense.*

Proof. Since the statements are local, we only consider the restriction of v to a large ball B_r. So for $x \in B_r$, we have

$$v(x) = \int_{\mathbb{R}^n} E(x - y) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \tilde{E}(x - y) f(y)\,\mathrm{d}y, \tag{144}$$

where $\tilde{E}(x) = \rho(x)E(x)$ with $\rho \in \mathscr{D}(B_{3r})$ satisfying $\rho \equiv 1$ in B_{2r} and $0 \leq \rho \leq 1$ everywhere. This implies that

$$\|v\|_{L^\infty(B_r)} \leq \|\tilde{E}\|_{L^1}\|f\|_{L^\infty} \leq cr^2\|f\|_{L^\infty}, \tag{145}$$

where c depends only on n. Now let us define $\tilde{E}_\varepsilon(x) = (1 - \rho(rx/\varepsilon))\tilde{E}(x)$, so that $\tilde{E}_\varepsilon \equiv 0$ in $B_{2\varepsilon}$ and $\tilde{E}_\varepsilon \equiv \tilde{E}$ in $\mathbb{R}^n \setminus \bar{B}_{3\varepsilon}$, and let $v_\varepsilon = \tilde{E}_\varepsilon * f$. Since $\tilde{E}_\varepsilon \in \mathscr{D}(\mathbb{R}^n)$, we have $v_\varepsilon \in \mathscr{D}(\mathbb{R}^n)$. Moreover, $v_\varepsilon \to v$ uniformly in B_r because

$$\|v - v_\varepsilon\|_{L^\infty(B_r)} \leq \|\tilde{E} - \tilde{E}_\varepsilon\|_{L^1}\|f\|_{L^\infty} \leq c\varepsilon^2\|f\|_{L^\infty}, \tag{146}$$

meaning that $v \in C(B_r)$. As r is arbitrary, we get $v \in C(\mathbb{R}^n)$.

As a preparation to showing that v is weakly differentiable, for $\varphi \in \mathscr{D}(\mathbb{R}^n)$ we compute

$$\begin{aligned}\int_{\mathbb{R}^n} \tilde{E}\partial_i\varphi &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon} \tilde{E}(x)\partial_i\varphi(x)\,\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0}\Big(\int_{\partial B_\varepsilon} E(x)\varphi(x)\frac{x_i}{|x|}\,\mathrm{d}^{n-1}x + \int_{\mathbb{R}^n\setminus B_\varepsilon}\partial_i\tilde{E}(x)\varphi(x)\,\mathrm{d}x\Big) \\ &= -\int_{\mathbb{R}^n}\partial_i\tilde{E}(x)\varphi(x)\,\mathrm{d}x,\end{aligned} \tag{147}$$

where in the last step we have used the fact that $\partial_i\tilde{E} \in L^1(\mathbb{R}^n)$. Now for $\varphi \in \mathscr{D}(B_r)$, we have

$$\begin{aligned}\int_{B_r} v\partial_i\varphi &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\tilde{E}(x-y)f(y)\,\mathrm{d}y\Big)\partial_i\varphi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\tilde{E}(x-y)\partial_i\varphi(x)\,\mathrm{d}x\Big)f(y)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\partial_i\tilde{E}(x-y)\varphi(x)\,\mathrm{d}x\Big)f(y)\,\mathrm{d}y \\ &= -\int_{B_r}\Big(\int_{\mathbb{R}^n}\partial_i\tilde{E}(x-y)f(y)\,\mathrm{d}y\Big)\varphi(x)\,\mathrm{d}x,\end{aligned} \tag{148}$$

where the use of the Fubini theorem is easily justified, and we have used the computation (147) in the third step. The expression

$$w(x) = \int_{\mathbb{R}^n}\partial_i\tilde{E}(x-y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n}\partial_i E(x-y)f(y)\,\mathrm{d}y, \tag{149}$$

defines a function $w \in L^\infty(B_r)$ because

$$\|w\|_{L^\infty(B_r)} \leq \|\partial_i\tilde{E}\|_{L^1}\|f\|_{L^\infty} \leq cr\|f\|_{L^\infty}, \tag{150}$$

where c depends only on n. Hence it follows from (148) that $\partial_i v = w$ in B_r in the weak sense. As r is arbitrary, the convolution (149) defines a locally bounded function w, and $\partial_i v = w$ in $\mathbb{R}^n$. Since $\partial_i\tilde{E}$ is locally integrable, the argument (146) can be adapted to show that $w \in C(\mathbb{R}^n)$ and therefore that $v \in C^1(\mathbb{R}^n)$.

It remains to prove that $\Delta v = f$ in the weak sense. To this end, recall that

$$\int_{\mathbb{R}^n} E(x-y)\Delta\varphi(x)\,\mathrm{d}x = \varphi(y), \tag{151}$$

for $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$. For $\varphi \in \mathscr{D}(B_r)$ this implies that

$$\begin{aligned}-\int_{B_r}\nabla v\cdot\nabla\varphi = \int_{B_r} v\Delta\varphi &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\tilde{E}(x-y)f(y)\,\mathrm{d}y\Big)\Delta\varphi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\tilde{E}(x-y)\Delta\varphi(x)\,\mathrm{d}x\Big)f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}E(x-y)\Delta\varphi(x)\,\mathrm{d}x\Big)f(y)\,\mathrm{d}y \\ &= \int_{B_r}\varphi(y)f(y)\,\mathrm{d}y,\end{aligned} \tag{152}$$

and since r is arbitrary, the proof is complete. □

We now start our investigation of the second derivatives of v.

Lemma 41. *Let*

$$w_\varepsilon(x) = \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \partial_i \partial_j E(x-y) f(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n, \quad \varepsilon > 0, \tag{153}$$

and suppose that $w_\varepsilon \to w$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *as* $\varepsilon \to 0$. *Then we have*

$$\partial_i \partial_j v = \frac{\delta_{ij}}{n} f + w \qquad \textit{in the weak sense.} \tag{154}$$

Proof. Let us do some preparatory computations. For $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $\varepsilon > 0$, we have

$$-\int_{\mathbb{R}^n \setminus B_\varepsilon} \partial_i E(x) \partial_j \varphi(x)\,\mathrm{d}x = \int_{\partial B_\varepsilon} \partial_i E(x) \varphi(x) \frac{x_j}{|x|}\,\mathrm{d}^{n-1}x + \int_{\mathbb{R}^n \setminus B_\varepsilon} \partial_i \partial_j E(x) \varphi(x)\,\mathrm{d}x. \tag{155}$$

Since $\partial_i E(x) = x_i/(|S^{n-1}||x|^n)$, the first term on the right hand side is

$$\int_{\partial B_\varepsilon} \partial_i E(x) \varphi(x) \frac{x_j}{|x|}\,\mathrm{d}^{n-1}x = \frac{1}{|S^{n-1}|\varepsilon^{n+1}} \int_{\partial B_\varepsilon} \varphi(x) x_i x_j\,\mathrm{d}^{n-1}x. \tag{156}$$

We have

$$\int_{\partial B_\varepsilon} x_i x_j\,\mathrm{d}^{n-1}x = 0, \qquad \text{for } i \neq j, \tag{157}$$

by the mean value property of harmonic functions[3], and

$$\int_{\partial B_\varepsilon} x_i^2\,\mathrm{d}^{n-1}x = \frac{1}{n} \int_{\partial B_\varepsilon} |x|^2\,\mathrm{d}^{n-1}x = \frac{|S^{n-1}|\varepsilon^{n+1}}{n}. \tag{158}$$

Therefore by using $\varphi(x) - \varphi(0) = O(\varepsilon)$ for $|x| = \varepsilon$, we get

$$\begin{aligned} \int_{\partial B_\varepsilon} \varphi(x) x_i x_j\,\mathrm{d}^{n-1}x &= \varphi(0) \int_{\partial B_\varepsilon} x_i x_j\,\mathrm{d}^{n-1}x + \int_{\partial B_\varepsilon} (\varphi(x) - \varphi(0)) x_i x_j\,\mathrm{d}^{n-1}x \\ &= \frac{|S^{n-1}|\varepsilon^{n+1}}{n} \delta_{ij} \varphi(0) + O(\varepsilon^{n+2}), \end{aligned} \tag{159}$$

leading to the formula

$$-\int_{\mathbb{R}^n \setminus B_\varepsilon} \partial_i E(x) \partial_j \varphi(x)\,\mathrm{d}x = \frac{\delta_{ij}}{n} \varphi(0) + \int_{\mathbb{R}^n \setminus B_\varepsilon} K(x) \varphi(x)\,\mathrm{d}x + O(\varepsilon). \tag{160}$$

[3] Thanks to Almaz Butaev for this argument

Now we fix a ball B_r with large radius, and let $\tilde{E}$ be as in the proof of Lemma 40. Then for $\varphi \in \mathscr{D}(B_r)$ we have

$$\begin{aligned}
\int_{B_r} v\partial_i\partial_j\varphi &= -\int_{B_r} \partial_i v\cdot\partial_j\varphi = -\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\partial_i\tilde{E}(x-y)f(y)\,\mathrm{d}y\Big)\partial_j\varphi(x)\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\partial_i\tilde{E}(x-y)\partial_j\varphi(x)\,\mathrm{d}x\Big)f(y)\,\mathrm{d}y\\
&= -\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n\setminus B_\varepsilon(y)}\partial_i E(x-y)\partial_j\varphi(x)\,\mathrm{d}x\Big)f(y)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^n}\frac{\delta_{ij}}{n}\varphi(y)f(y)\,\mathrm{d}y + \lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n\setminus B_\varepsilon(y)}K(x-y)\varphi(x)\,\mathrm{d}x\Big)f(y)\,\mathrm{d}y \qquad (161)\\
&= \frac{\delta_{ij}}{n}\int_{\mathbb{R}^n} f(y)\varphi(y)\,\mathrm{d}y + \lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}K(x-y)f(y)\,\mathrm{d}y\Big)\varphi(x)\,\mathrm{d}x\\
&= \frac{\delta_{ij}}{n}\int_{\mathbb{R}^n} f(y)\varphi(y)\,\mathrm{d}y + \lim_{\varepsilon\to0}\int_{\mathbb{R}^n} w_\varepsilon(x)\varphi(x)\,\mathrm{d}x\\
&= \frac{\delta_{ij}}{n}\int_{\mathbb{R}^n} f(y)\varphi(y)\,\mathrm{d}y + \int_{\mathbb{R}^n} w(x)\varphi(x)\,\mathrm{d}x,
\end{aligned}$$

where we have used (160) in the fourth step. The lemma has been proven. □

We see that an important role is played by the kernel $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ given by

$$K(x) = \partial_i\partial_j E(x) \equiv \frac{\delta_{ij}|x|^2 - nx_ix_j}{|x|^{n+2}}. \qquad (162)$$

The following properties will be useful.

i) Bounds: $|K(x)| \leq c|x|^{-n}$ and $|\nabla K(x)| \leq c|x|^{-n-1}$.
ii) Cancellation property: $\int_{B_R\setminus B_r} K = 0$ for any $0 < r < R < \infty$.

Proving the bounds is straightforward, and the cancellation property may be shown by applying the mean value property to the function $\delta_{ij}|x|^2 - nx_ix_j$.

We want to get some insight on when the function w from the preceding lemma is well-defined. By a change of variables, we have

$$w_\varepsilon(x) = \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} K(x-y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n\setminus B_\varepsilon} K(z)f(x-z)\,\mathrm{d}z. \qquad (163)$$

As before, we confine x to a large fixed ball B_r. If $R > 0$ is sufficiently large, it holds that

$$\int_{\mathbb{R}^n\setminus B_\varepsilon} K(z)f(x-z)\,\mathrm{d}z = \int_{B_R\setminus B_\varepsilon} K(z)f(x-z)\,\mathrm{d}z, \qquad (164)$$

and then the cancellation property implies that

$$w_\varepsilon(x) = \int_{B_R\setminus B_\varepsilon} K(z)f(x-z)\,\mathrm{d}z = \int_{B_R\setminus B_\varepsilon} K(z)(f(x-z)-f(x))\,\mathrm{d}z. \qquad (165)$$

Introducing the *modulus of continuity*

$$\omega(f,t) = \sup_{|h|\leq t}\|\Delta_h f\|_{L^\infty(\mathbb{R}^n)} \equiv \sup_{|x-y|\leq t}|f(x)-f(y)|, \qquad (t>0), \qquad (166)$$

we get a control on $|w_\varepsilon(x)|$ as

$$|w_\varepsilon(x)| \leq \int_{B_R\setminus B_\varepsilon} |K(z)|\,\omega(|z|)\,\mathrm{d}z \leq c\int_\varepsilon^R t^{-n}\omega(f,t)t^{n-1}\,\mathrm{d}t = c\int_\varepsilon^R \frac{\omega(f,t)}{t}\,\mathrm{d}t. \qquad (167)$$

So if

$$\int_0^1 \frac{\omega(f,t)}{t}\,\mathrm{d}t < \infty. \tag{168}$$

then the pointwise limit $w(x) = \lim_{\varepsilon\to 0} w_\varepsilon(x)$ exists, and in addition, $\|w_\varepsilon\|_{L^\infty(B_r)} \leq c$ with a constant $c = c(r)$ independent of $\varepsilon > 0$. Hence $w_\varepsilon \to w$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\varepsilon \to 0$. The condition (168) is called the *Dini condition*, and functions satisfying it are called *Dini continuous*. An example of Dini continuity is Hölder continuity, where we have $\omega(r) = r^\alpha$ with $\alpha > 0$.

Now that we have a function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying $\partial_i\partial_j v = \delta_{ij}n^{-1}f + w$, in order to get $v \in C^2$ we need to show that w is continuous.

Theorem 42. *Let f be Dini continuous and compactly supported in $\mathbb{R}^n$. Then $w \in C(\mathbb{R}^n)$.*

Proof. Let $x, x' \in \mathbb{R}^n$ with $\delta = |x - x'| > 0$ small. Then we have

$$\begin{aligned} |w(x) - w(x')| \leq \Big|\int_{|z|<3\delta} K(z)f(x-z)\,\mathrm{d}z\Big| + \Big|\int_{|z|<3\delta} K(z)f(x'-z)\,\mathrm{d}z\Big| \\ + \Big|\int_{|z|>3\delta} K(z)(f(x-z) - f(x'-z))\,\mathrm{d}z\Big|, \end{aligned} \tag{169}$$

and the first two terms on the right hand side are bounded by a constant multiple of

$$\int_0^{3\delta} \frac{\omega(f,t)}{t}\,\mathrm{d}t, \tag{170}$$

which goes to 0 as $\delta \to 0$. It remains to bound the second term, which we denote by I henceforth:

$$I = \int_{|z|>3\delta} K(z)(f(x-z) - f(x'-z))\,\mathrm{d}z. \tag{171}$$

By making use of the cancellation property of K, we derive

$$\begin{aligned} \int_{|z|>3\delta} K(z)f(x-z)\,\mathrm{d}z &= \int_{|z|>3\delta} K(z)(f(x-z) - f(x))\,\mathrm{d}z \\ &= \int_{\mathbb{R}^n} K_\delta(x-y)(f(y) - f(x))\,\mathrm{d}y, \end{aligned} \tag{172}$$

where $K_\delta(z) = K(z)$ for $|z| > 3\delta$ and $K_\delta(z) = 0$ otherwise. The same argument gives

$$\begin{aligned} \int_{|z|>3\delta} K(z)f(x'-z)\,\mathrm{d}z &= \int_{|z|>3\delta} K(z)(f(x'-z) - f(x))\,\mathrm{d}z \\ &= \int_{\mathbb{R}^n} K_\delta(x'-y)(f(y) - f(x))\,\mathrm{d}y, \end{aligned} \tag{173}$$

and hence, with $R > 0$ is large enough so that $\mathrm{supp} f \subset B_R$,

$$\begin{aligned} I &= \int_{\mathbb{R}^n} (K_\delta(x-y) - K_\delta(x'-y))(f(y) - f(x))\,\mathrm{d}y \\ &= \int_{B_R} (K_\delta(x-y) - K_\delta(x'-y))(f(y) - f(x))\,\mathrm{d}y. \end{aligned} \tag{174}$$

Observe that $K_\delta(x-y) = 0$ and $K_\delta(x'-y) = 0$ when $y \in B_{2\delta}(x)$, and that $K_\delta(x-y) = K(x-y)$ and $K_\delta(x'-y) = K(x'-y)$ when $y \notin B_{4\delta}(x)$. So for $y \in B_{4\delta}(x)$, we use the bound

$$|K_\delta(x-y) - K_\delta(x'-y)| \leq |K_\delta(x-y)| + |K_\delta(x'-y)| \leq c\delta^{-n}, \tag{175}$$

and for $y \notin B_{4\delta}(x)$, we use

$$\begin{aligned} |K_\delta(x-y) - K_\delta(x'-y)| = |K(x-y) - K(x'-y)| &\leq \max_{\xi\in[x,x']} |\nabla K(\xi - y)| \cdot |x - x'| \\ &\leq c\delta \max_{\xi\in[x,x']} |\xi - y|^{-n-1} \leq c\delta|x-y|^{-n-1}, \end{aligned} \tag{176}$$

because $|x - y| \leq \frac{4}{3}|\xi - y|$ for $\xi \in [x, x']$ and $y \notin B_{4\delta}(x)$. Note that we are using the letter c to denote different constants at its different occurrences. The bounds we have yield

$$\begin{aligned} |I| &\leq c\delta^{-n} \int_{|x-y|<4\delta} \omega(f, |x-y|)\,\mathrm{d}y + c\delta \int_{4\delta<|x-y|<R} \frac{\omega(f,|x-y|)}{|x-y|^{n+1}}\,\mathrm{d}y \\ &\leq c\delta^{-n} \int_0^{4\delta} t^{n-1}\omega(f,t)\,\mathrm{d}t + c\delta \int_{4\delta}^R \frac{\omega(f,t)}{t^2}\,\mathrm{d}t \\ &\leq c \int_0^{4\delta} \frac{\omega(f,t)}{t}\,\mathrm{d}t + c\delta \int_\delta^R \frac{\omega(f,t)}{t^2}\,\mathrm{d}t. \end{aligned} \tag{177}$$

It is clear that the first term on the right hand side goes to 0 as $\delta \to 0$. To see that the second term vanishes as $\delta \to 0$, write[4]

$$\delta \int_\delta^R \frac{\omega(f,t)}{t^2}\,\mathrm{d}t \leq \delta \int_\delta^\rho \frac{\omega(f,t)}{t^2}\,\mathrm{d}t + \delta \int_\rho^R \frac{\omega(f,t)}{t^2}\,\mathrm{d}t \leq \int_0^\rho \frac{\omega(f,t)}{t}\,\mathrm{d}t + \frac{\delta}{\rho}\int_\rho^R \frac{\omega(f,t)}{t}\,\mathrm{d}t, \tag{178}$$

which is valid for any $\delta < \rho < R$. For any given $\varepsilon > 0$, we choose $\rho > 0$ so small that the first term on the right hand side is smaller than ε. Then we choose $\delta > 0$ so small that the second term is smaller than ε. The proof is established. □

Corollary 43. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a weak solution of $\Delta u = f$, where f is Dini continuous in Ω. Then $u \in C^2(\Omega)$.*

Proof. Let B be a (nontrivial) ball whose closure is contained in Ω, and let $\phi \in \mathscr{D}(\Omega)$ be such that $\phi \equiv 1$ in B. Then ϕf is a compactly supported Dini function because

$$\omega(\phi f, t) \leq \|\phi\|_{L^\infty(\mathbb{R}^n)}\omega(f,t) + \|f\|_{L^\infty(\mathrm{supp}\phi)}\omega(\phi,t). \tag{179}$$

Hence the Newtonian potential $v = E * (\phi f)$ satisfies $v \in C^2(\mathbb{R}^n)$ and $\Delta v = \phi f$ in $\mathbb{R}^n$. In particular, $\Delta v = f$ in B, which means that $u - v$ is a weak solution to $\Delta(u - v) = 0$ in B, and so $u - v$ is analytic in B. This shows that $u|_B \in C^2(B)$. As B is arbitrary, we conclude that $u \in C^2(\Omega)$. □

[4]Thanks to Ben Landon for this argument