MATH 580 TAKE HOME MIDTERM EXAM 2

DUE MONDAY DECEMBER 2

1. In this exercise we continue our study of Sobolev spaces on the interval $I = (0, 1)$. Let $1 \leq p < \infty$, and recall the norm

$$\|u\|_{1,p} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{1/p},$$

for $u \in C^1(\bar{I})$. Then we define $H^{1,p}(I)$ to be the completion of $C^1(\bar{I})$ with respect to the norm $\|\cdot\|_{1,p}$, and let

$$W^{1,p}(I) = \{u \in L^p(I) : u' \in L^p(I)\},$$

where u' is understood in the weak sense. Prove the followings.
(a) The *Meyers-Serrin theorem*: $H^{1,p}(I) = W^{1,p}(I)$.
(b) The space $\{u \in C^1(\bar{I}) : u'(0) = u'(1) = 0\}$ is dense in $W^{1,p}(I)$.
(c) The *Rellich-Kondrashov theorem*: The embedding $W^{1,p}(I) \hookrightarrow L^p(I)$ is compact.

2. For a function u on $A \subset \mathbb{R}^n$, and $\alpha > 0$, let

$$|u|_{\mathrm{Lip}\alpha} = \sup_{x,y \in A} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

and define the *Hölder space* $C^{k,\alpha}(A)$ as the space of functions $u \in C^k(A)$ for which

$$\|u\|_{C^{k,\alpha}(A)} = \sum_{|\beta| \leq k} \|\partial^\beta u\|_{C(A)} + \sum_{|\beta| = k} |\partial^\beta u|_{\mathrm{Lip}\alpha},$$

is finite. In the following, we fix $0 < \alpha < 1$.
(a) For a compactly supported function f, let

$$(Tf)(x) = \lim_{\varepsilon \to 0} \int\limits_{\{|x-y| > \varepsilon\}} K(x - y) f(y)\, \mathrm{d}y,$$

where the kernel $K \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfies
- $|K(x)| \leq c|x|^{-n}$ and $|\nabla K(x)| \leq c|x|^{1-n}$ for some constant c,
- $\int_{B_R \setminus B_r} K(x)\, \mathrm{d}x = 0$ for $0 < r < R < \infty$.

We showed in class that Tf is well-defined and continuous if f is Dini continuous. Prove that for any $0 < r < R < \infty$ there exists a constant C such that

$$\|Tf\|_{C^{0,\alpha}(B_r)} \leq C \|f\|_{C^{0,\alpha}(B_R)},$$

for all $f \in C^{0,\alpha}(\mathbb{R}^n)$ with $\mathrm{supp} f \subset B_R$.

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(b) Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in H^1_{\text{loc}}(\Omega)$ be a weak solution of $\Delta u = f$ in Ω, with $f \in C^{0,\alpha}(\Omega)$. Show that $u \in C^{2,\alpha}(K)$ for any compact $K \subset \Omega$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let

$$a(u,v) = \int_\Omega (a_{ij}\partial_i u \partial_j v + cuv),$$

where the repeated indices are summer over, and the coefficients a_{ij} and c are smooth functions on $\overline{\Omega}$, with a_{ij} satisfying the uniform ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \xi \in \mathbb{R}^n, \quad x \in \overline{\Omega},$$

for some constant $\lambda > 0$.

(a) Let $u \in H^1_0(\Omega)$ and $f \in H^k(\Omega)$ with $k \geq 0$ satisfy $a(u,v) = \int_\Omega fv$ for all $v \in H^1_0(\Omega)$. Prove that $u \in H^{k+2}_{\text{loc}}(\Omega)$. Sketch the ideas on how to prove $u \in H^{k+2}(\Omega)$.

(b) Let $u \in H^1(\Omega)$ and $f \in H^k(\Omega)$ with $k \geq 0$ satisfy $a(u,v) = \int_\Omega fv$ for all $v \in H^1(\Omega)$. Prove that $u \in H^{k+2}_{\text{loc}}(\Omega)$. Sketch the ideas on how to prove $u \in H^{k+2}(\Omega)$.

4. Let the bilinear form a be as in the preceding exercise.

- (Dirichlet case) We define the map $\tilde{A}_D : H^1_0(\Omega) \to [H^1_0(\Omega)]'$ by $\langle \tilde{A}_D u, v\rangle = a(u,v)$ for $u, v \in H^1_0(\Omega)$, and then we let A_D be the unbounded operator in $L^2(\Omega)$ that is given by the restriction of $\tilde{A}_D$ on $L^2(\Omega)$, i.e., $A_D u = \tilde{A}_D u$ for $u \in \text{Dom}(A_D)$ where $\text{Dom}(A_D) = \{u \in H^1_0(\Omega) : \tilde{A}_D u \in L^2(\Omega)\}$.
- (Neumann case) Similarly, we define the operator $\tilde{A}_N : H^1(\Omega) \to [H^1(\Omega)]'$ by $\langle \tilde{A}_N u, v\rangle = a(u,v)$ for $u, v \in H^1(\Omega)$, and then we let A_N be the unbounded operator in $L^2(\Omega)$ that is given by the restriction of $\tilde{A}_N$ on $L^2(\Omega)$, i.e., $A_N u = \tilde{A}_N u$ for $u \in \text{Dom}(A_N)$ where $\text{Dom}(A_N) = \{u \in H^1(\Omega) : \tilde{A}_N u \in L^2(\Omega)\}$.

Consider the eigenvalue problem

$$Au = \lambda u,$$

on a bounded C^1 domain $\Omega \subset \mathbb{R}^n$, where A is either A_D or A_N. Prove the followings, by using the spectral theorem for compact self-adjoint positive operators where possible.

(a) The eigenvalues $\{\lambda_k\}$ are countable and real, and that $\lambda_k \to \infty$ as $k \to \infty$. Each eigenvalue has a finite multiplicity.

(b) The eigenfunctions $\{u_k\}$ form a complete orthonormal system in $L^2(\Omega)$.

(c) In the Dirichlet case, the system $\{u_k\}$ is complete and orthogonal in $H^1_0(\Omega)$, with respect to the inner product $a(u,v) + t\int_\Omega uv$, where t is a suitably chosen constant. The same holds for the Neumann case, with $H^1_0(\Omega)$ replaced by $H^1(\Omega)$.

(d) The eigenfunctions are smooth in Ω, and are smooth up to the boundary if $\partial\Omega$ is smooth.

(e) Explicitly compute the eigenvalues and eigenfunctions of the Laplacian with the homogeneous Dirichlet boundary condition on the rectangle $\Omega = (0,a) \times (0,b)$. Make sure that you don't miss any eigenfunction, i.e., prove that under suitable scaling, the functions you computed form a complete orthonormal system in $L^2(\Omega)$.