MATH 580 TAKE HOME MIDTERM EXAM 1

DUE WEDNESDAY OCTOBER 23

1. Let Ω be a domain, and let $\Sigma = \partial\Omega \cap B$ be a smooth and nonempty portion of the boundary, where B is an open ball. Let $u \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma)$ satisfy $\Delta u = 0$ in Ω and $u = \partial_\nu u = 0$ on Σ. Show that u is identically zero in Ω.
2. Consider the problem of minimizing the energy
$$Q(u) = \int_I \left(1 + |u'(x)|^2\right)^{\frac{1}{4}} \,\mathrm{d}x,$$
for all $u \in C^1(I) \cap C(\bar{I})$ satisfying $u(0) = 0$ and $u(1) = 1$, where $I = (0,1)$. Show that the infimum of Q over the admissible functions is 1, but this value is not assumed by any admissible function.
3. Consider the Dirichlet problem on the unit disk $\mathbb{D} \subset \mathbb{R}^2$ with the homogeneous Dirichlet boundary condition. The solution is $u \equiv 0$, which also minimizes the Dirichlet energy
$$E(u) = \int_{\mathbb{D}} |\nabla u|^2.$$
Construct a sequence $\{u_k\} \subset C(\overline{\mathbb{D}})$ of functions satisfying all of the following conditions.
   - u_k is piecewise smooth and $u_k|_{\partial\mathbb{D}} = 0$ for all k,
   - $E(u_k) \to 0$ as $k \to \infty$, and
   - u_k as $k \to \infty$ diverges in a set that is dense in $\mathbb{D}$.

   Then show that $u_k \to 0$ in $H^1(\mathbb{D})$.
4. Exhibit a sequence $\{v_k\} \subset \tilde{C}^1(\mathbb{D})$ that is Cauchy with respect to the H^1-norm, whose limit is not essentially bounded.
5. In this exercise we will study Sobolev spaces on the interval $I = (0,1)$. Let $1 \leq p < \infty$, and define the norm
$$\|u\|_{1,p} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{1/p},$$
for $u \in C^1(\bar{I})$. Let $H^{1,p}(I)$ be the completion of $C^1(\bar{I})$ with respect to the norm $\|\cdot\|_{1,p}$.
a) Show that there is a continuous injection of $H^{1,p}(I)$ into $L^p(I)$.
b) Prove the *Sobolev inequality*
$$\|u\|_{L^\infty} \leq 2^{1-1/p}\|u\|_{1,p}, \qquad u \in H^{1,p}(I).$$
c) Since $H^{1,p}(I)$ is a subspace of $L^p(I)$, an element of $H^{1,p}(I)$ is an equivalence class of functions that differ on sets of measure zero. So one can change the values of a function on a set of measure zero, and it would still correspond to the same element

Date: Fall 2013.

in $H^{1,p}(I)$. Make sense of, and prove the statement that the elements of $H^{1,p}(I)$ are continuous functions.

d) Prove the *Friedrichs inequality*

$$\|u\|_{L^p}^p \leq 2^{p-1}\|u'\|_{L^p}^p + 2^{p-1}|u(\xi)|^p, \qquad u \in H^{1,p}(I), \quad \xi \in [0,1].$$

In particular, make sense of the derivative u' appearing in the right hand side.

e) Let $H_0^{1,p}(I)$ be the closure of $C_c^1(I)$ in $H^{1,p}(I)$. Show that

$$H_0^{1,p}(I) = \{u \in H^{1,p}(I) : u(0) = u(1) = 0\}.$$

f) Prove the *Poincaré inequality*

$$\|u\|_{L^p}^p \leq 2^{p-1}\|u'\|_{L^p}^p + 2^{p-1}\Big|\int_I u\Big|^p, \qquad u \in H^{1,p}(I).$$

6. a) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $g \in H^1(\Omega)$. Give a detailed proof of the fact that there exists a unique $u \in H^1(\Omega)$ satisfying $u - g \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla v = 0 \qquad \text{for all} \quad v \in \mathscr{D}(\Omega).$$

b) In the context of a), let us denote the map that sends g to u by

$$S : g \mapsto u : H^1(\Omega) \to H^1(\Omega).$$

Show that S is linear and is Lipschitz continuous in the sense that there is a constant c such that

$$\|S(g_1) - S(g_2)\|_{H^1} \leq c\|g_1 - g_2\|_{H^1}, \qquad g_1, g_2 \in H^1(\Omega).$$

c) Exhibit an example of an unbounded domain Ω and a sequence $\{u_k\} \subset H_0^1(\Omega)$ such that $\{u_k\}$ minimizes the Dirichlet energy on Ω and that it is not Cauchy with respect to the H^1-norm.