SPECTRAL PROPERTIES OF THE LAPLACIAN ON BOUNDED DOMAINS

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Abstract. After establishing discrete spectra for a large class of elliptic operators, we present some fundamental spectral properties of the Dirichlet and Neumann Laplace operators on bounded domains, including eigenvalue comparison theorems, Weyl's asymptotic law, and Courant's nodal domain theorem. *Note*: This is an incomplete draft.

Contents

1. Introduction

In these notes, we will be concerned with the *eigenvalue problem*

$$-\Delta u = \lambda u, \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, with either the Dirichlet $u = 0$ or the Neumann $\partial_\nu u = 0$ condition on the boundary $\partial\Omega$. The unknown in the problem is the pair (u, λ) where u is a function and λ is a number. If (u, λ) is a solution then u is called an *eigenfunction*, and λ is called the *eigenvalue* associated to u. Let us note the following.

- Since the right hand side involves λu, the problem is *not* linear.
- If (u, λ) is a solution then so is $(\alpha u, \lambda)$ for any number α.
- We exclude the trivial solution $u = 0$ from all considerations.

We have studied the problem $-\Delta u + tu = f$ with the Dirichlet or Neumann boundary conditions, where $t \in \mathbb{R}$ and $f \in L^2(\Omega)$ are given. Since (1) is equivalent to $-\Delta u + tu = 0$ with $t = -\lambda$, we can give the following weak formulation for (1). Let V be either $H^1_0(\Omega)$ or $H^1(\Omega)$, depending on the boundary condition we wish to impose. Then the problem is to find $u \in V$ and $\lambda \in \mathbb{R}$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega uv \qquad \text{for all } v \in V. \tag{2}$$

We know that a unique weak solution $u \in V$ to $-\Delta u + tu = f$ exists if t is larger than a certain threshold value t_0 that depends on the type of the boundary condition and the geometry of the domain Ω. This shows that if (2) has a nontrivial solution then $-\lambda \leq t_0$, that is, λ cannot be less than $-t_0$. If $u \in V$ satisfies (2) then the regularity results imply that $u \in C^\omega(\Omega)$, and so in particular u is a classical solution of (1) in Ω, and moreover that u satisfies the desired boundary condition in the classical sense provided the boundary is regular enough.

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Now we want to write (2) as an abstract operator eigenvalue problem. We introduce the linear operator $A : V \to V'$ and the bilinear form $a : V \times V \to \mathbb{R}$ by[1]

$$\langle Au, v\rangle = a(u, v) = \int_\Omega \nabla u \cdot \nabla v, \qquad u, v \in V, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between V' and V. Recall that a is continuous

$$|a(u, v)| \leq \|u\|_{H^1}\|v\|_{H^1}, \qquad u, v \in V, \tag{4}$$

symmetric

$$a(u, v) = a(v, u), \qquad u, v \in V, \tag{5}$$

and satisfies

$$a(u, u) + t\langle u, u\rangle_{L^2} \geq \alpha\|u\|^2_{H^1}, \qquad u \in V, \tag{6}$$

for all $t > t_0$, with $\alpha > 0$ possibly depending on t. The operator A is called the *energy extension* of $-\Delta$ with the given boundary condition, in the sense that it is an extension of the classical Laplacian acting on a dense subset of V. We can check that it is bounded:

$$\|Au\|_{V'} = \sup_{v\in V} \frac{\langle Au, v\rangle}{\|v\|_{H^1}} = \sup_{v\in V} \frac{a(u, v)}{\|v\|_{H^1}} \leq \|u\|_{H^1}. \tag{7}$$

In terms of the operator A, the problem (2) can be written as

$$Au = \lambda Ju, \tag{8}$$

where the inclusion map $J : L^2(\Omega) \to V'$ is defined by

$$\langle Jf, v\rangle = \int_\Omega fv, \qquad v \in V. \tag{9}$$

Obviously, J is injective because $Jf = 0$ implies $f = 0$ for $f \in L^2(\Omega)$ by the du Bois-Reymond lemma. It is also continuous:

$$\|Jf\|_{V'} = \sup_{v\in V} \frac{\langle Jf, v\rangle}{\|v\|_{H^1}} \leq \sup_{v\in V} \frac{\|f\|_{L^2}\|v\|_{L^2}}{\|v\|_{H^1}} \leq \|f\|_{L^2(\Omega)}, \tag{10}$$

and hence J defines a continuous embedding of $L^2(\Omega)$ into V'. In what follows we will identify $L^2(\Omega)$ with a subspace of V' through J. So for instance, we write (8) simply as

$$Au = \lambda u. \tag{11}$$

By the Riesz representation theorem, $A + tI$ is invertible for $t > t_0$, and $(A + tI)^{-1} : V' \to V$ is bounded. In what follows, we fix some $t > t_0$. Then adding tu to both sides of (11), and applying $(A + tI)^{-1}$, we get

$$u = (t + \lambda)(A + tI)^{-1}u. \tag{12}$$

At this point, we introduce the *resolvent*[2]

$$R_t = (A + tI)^{-1}|_{L^2(\Omega)} : L^2(\Omega) \to L^2(\Omega), \tag{13}$$

which is the restriction of $(A + tI)^{-1}$ to $L^2(\Omega)$. Hence if $u \in V$ and $\lambda \in \mathbb{R}$ satisfy (11), then

$$(t + \lambda)R_t u = u. \tag{14}$$

Conversely, if $u \in L^2(\Omega)$ and $\lambda \in \mathbb{R}$ satisfy (14), then by applying $A + tI$ on both sides, we derive (11), proving the equivalence of the two formulations.

Let us derive some straightforward properties of the resolvent.

[1] Note that A in these notes corresponds to $\tilde{A}$ from class. In class we denoted by A the Friedrichs extension, but I realized later that we don't really need it and removing it makes the presentation a bit simpler.

[2] The usual definition is $(A - tI)^{-1}$, but we are using the plus sign for convenience.

- The resolvent is bounded as an operator $R_t : L^2(\Omega) \to V$, because
$$\|R_t f\|_V \leq c\|f\|_{V'} \leq c\|f\|_{L^2(\Omega)}, \tag{15}$$
where the constant c may have different values at its different occurrences.
- The resolvent is *positive*, in the sense that $\langle R_t f, f\rangle > 0$ for $f \neq 0$, where $\langle\cdot,\cdot\rangle$ is the L^2 inner product on Ω. To see this, let $f \in L^2(\Omega)$ and let $u = R_t f$, so that $f = (A+tI)u$. Then the strict coercivity property (6) gives
$$\langle R_t f, f\rangle = \langle u, (A+tI)u\rangle = a(u,u) + t\langle u,u\rangle \geq \alpha\|u\|^2_{H^1}. \tag{16}$$
- The resolvent is injective: If $R_t f = 0$ then $f = 0$.
- The resolvent is *symmetric*, in the sense that $\langle R_t f, g\rangle = \langle f, R_t g\rangle$ for $f, g \in L^2(\Omega)$. With $u = R_t f$ and $v = R_t g$, we have
$$\langle R_t f, g\rangle = \langle u, (A+tI)v\rangle = a(u,v) + t\langle u,v\rangle, \tag{17}$$
which clearly shows the claim.
- A function $u \in L^2(\Omega)$ is in the range of R_t if and only if $Au \in L^2(\Omega)$. To show this, first let u be such that $Au \in L^2(\Omega)$. Then $(A+tI)u \in L^2(\Omega)$, hence $u = R_t(A+tI)u$, which means that $u \in \mathrm{Ran}R_t$. Second, let $u \in \mathrm{Ran}R_t$, i.e., let $u = R_t f$ for some $f \in L^2(\Omega)$. It is obvious that $f = (A+tI)u$. From this, we have $Au = f - tu \in L^2(\Omega)$.

Apart from these simple properties, a crucial property we would like to have for the resolvent is *compactness*. Recall *Rellich's lemma*, which says that if Ω is a bounded domain then the unit ball in $H^1_0(\Omega)$ is relatively compact in $L^2(\Omega)$, i.e., the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Since in the Dirichlet case, i.e., when $V = H^1_0(\Omega)$, the resolvent as a map $R_t : L^2(\Omega) \to H^1_0(\Omega)$ is bounded, the image of the unit ball of $L^2(\Omega)$ under R_t is a bounded set in $H^1_0(\Omega)$. Hence by Rellich's lemma this image is relatively compact in $L^2(\Omega)$, showing that *the resolvent as a map $R_t : L^2(\Omega) \to L^2(\Omega)$ is a compact operator*.

Therefore in the Dirichlet case, boundedness of Ω is sufficient for the compact resolvent. In the Neumann case, it is known that the resolvent is not compact without additional assumptions on the boundary regularity of Ω.

Definition 1. An open set $\Omega \subset \mathbb{R}^n$ is said to have the *H^1-extension property*, if there exists a bounded linear map $E : H^1(\Omega) \to H^1(\mathbb{R}^n)$.

For example, it is known that Lipschitz domains have the H^1-extension property.

Exercise 2. Show that in the Neumann case, if $\Omega \subset \mathbb{R}^n$ is bounded and has the H^1-extension property, then the resolvent as a map $R_t : L^2(\Omega) \to L^2(\Omega)$ is compact.

Our strategy to solve the Laplace eigenvalue problem (11) is through the equivalent formulation (14) in terms of the resolvent. The main feature that makes this formulation attractive is the fact that the resolvent is compact under some very mild assumptions on Ω.

2. Spectral theory of compact self-adjoint operators

In this section, we will prove the spectral theorem for compact, symmetric, positive operators on a real Hilbert space H. This theorem will then be applied to the resolvent in the next section. We start with some preliminary results.

Lemma 3. *Let $B : H \to H$ be a bounded symmetric operator. Then we have the following.*

a) If B is positive, and if $u \in H \setminus \{0\}$ and $\lambda \in \mathbb{R}$ satisfy $Bu = \lambda u$, then $\lambda > 0$.
b) If $u, v \in H \setminus \{0\}$ and $\lambda, \mu \in \mathbb{R}$ satisfy $Bu = \lambda u$, $Bv = \mu v$, and $\lambda \neq \mu$, then $\langle u, v\rangle = 0$.
c) Let $\{u_n\}$ be a complete orthonormal basis of H, such that $Bu_n = \lambda_n u_n$ for each n. Suppose that $u \in H$ satisfies $Bu = \lambda u$ with $\lambda \notin \{\lambda_n\}$. Then $u = 0$.

Proof. a) Recall that positivity of $B : H \to H$ means that $\langle Bv, v\rangle > 0$ for $v \in H \setminus \{0\}$. This implies that $\lambda\langle u, u\rangle = \langle Bu, u\rangle > 0$, hence $\lambda > 0$.

b) By symmetry, we have $\lambda\langle u, v\rangle = \langle Bu, v\rangle = \langle u, Bv\rangle = \mu\langle u, v\rangle$, hence $(\lambda - \mu)\langle u, v\rangle = 0$.

c) Recall that completeness of $\{u_n\}$ means that $v \in H$ and $\langle v, u_n\rangle = 0$ for all n imply $v = 0$. But since $\lambda \notin \{\lambda_n\}$, Part b) shows that $\langle u, u_n\rangle = 0$ for all n. □

Lemma 4. *Let $K : H \to H$ be a compact operator. Then we have the following.*

a) Each eigenvalue has a finite multiplicity.

b) The only possible accumulation point of the set of eigenvalues is 0.

c) If K is symmetric and positive, the norm of K is characterized by

$$\|K\| = \sup_{u\in H} \frac{\langle Ku, u\rangle}{\|u\|^2}. \tag{18}$$

Proof. a) Suppose that there is an eigenvalue μ with infinite multiplicity, i.e., let $\{v_k\}$ be a countable orthonormal set of vectors satisfying

$$Kv_k = \mu v_k, \qquad k = 1, 2, \ldots. \tag{19}$$

We can interpret the latter as $\{v_k\}$ being the image of the set $\{\mu^{-1}v_k\}$ under K. Since $\{\mu^{-1}v_k\}$ is a bounded set, the set $\{v_k\}$ is relatively compact, meaning that after passing to a subsequence, v_k converges to some element of H. However, we have $\|v_j - v_k\|^2 = 2$ for $j \neq k$, which leads to a contradiction.

b) If $\alpha \neq 0$ is an accumulation point of eigenvalues, then there exists a countable orthonormal set $\{v_k\}$ of eigenvectors with corresponding eigenvalues μ_k satisfying $\inf_k |\mu_k| > 0$. The latter condition ensures that $\{\mu_k^{-1}v_k\}$ is a bounded set, and the argument we have used in Part a) leads to a contradiction.

c) For any $u \in H$ we have

$$|\langle Ku, u\rangle| \leq \|Ku\|\|u\| \leq \|K\|\|u\|^2, \tag{20}$$

which shows that

$$\mu = \sup_{u\in H} \frac{|\langle Ku, u\rangle|}{\|u\|^2} \leq \|K\|. \tag{21}$$

On the other hand, since K is symmetric and positive, we have

$$0 \leq \langle K(u + tv), u + tv\rangle = \langle Ku, u\rangle + 2t\langle Ku, v\rangle + t^2\langle Kv, v\rangle, \tag{22}$$

for all $t \in \mathbb{R}$, which leads to the Cauchy-Bunyakowsky-Schwarz inequality

$$|\langle Ku, v\rangle|^2 \leq \langle Ku, u\rangle\langle Kv, v\rangle, \qquad u, v \in H. \tag{23}$$

We use to to derive

$$\langle Ku, Ku\rangle \leq \langle Ku, u\rangle^{\frac{1}{2}} \cdot \langle K^2u, Ku\rangle^{\frac{1}{2}} \leq \mu^{\frac{1}{2}}\|u\| \cdot \mu^{\frac{1}{2}}\|Ku\|, \tag{24}$$

which implies that $\|Ku\| \leq \mu\|u\|$ for all $u \in H$, hence $\|K\| \leq \mu$. □

Remark 5. The ratio $\langle Ku, u\rangle/\|u\|^2$ is called the *Rayleigh quotient* of u.

We are ready to prove the main result of this section.

Theorem 6. *Let $K : H \to H$ be a compact, symmetric and positive operator. Then K admits a countable set of eigenvectors $\{u_n\}$ that forms a complete orthonormal system in H. Moreover, denoting the corresponding eigenvalues by $\mu_1 \geq \mu_2 \geq \ldots$, we have the variational characterization*

$$\mu_n = \sup_{u\in H_{n-1}} \frac{\|Ku\|}{\|u\|} = \sup_{u\in H_{n-1}} \frac{\langle Ku, u\rangle}{\|u\|^2}, \qquad \textit{for } n = 1, 2, \ldots, \tag{25}$$

where H_{n-1} is the orthogonal complement of $\operatorname{span}\{u_1, \ldots, u_{n-1}\}$ *in H.*

Proof. Let $\{v_i\}$ be a sequence in H such that $\|v_i\| = 1$ and

$$\langle Kv_i, v_i\rangle \to \mu_1 := \|K\| > 0. \tag{26}$$

Since the set $\{Kv_i\}$ is relatively compact, possibly passing to a subsequence, we can assume that $Kv_i \to w$ in H. On the other hand, we have

$$\begin{aligned} 0 \le \|Kv_i - \mu_1 v_i\|^2 &= \|Kv_i\|^2 + \mu_1^2 - 2\mu_1\langle Kv_i, v_i\rangle \\ &\le 2\mu_1\left(\mu_1 - \langle Kv_i, v_i\rangle\right) \to 0, \qquad \text{as } i \to \infty, \end{aligned} \tag{27}$$

which shows that $\mu_1 v_i \to w$, that is, $v_i \to u_1 := \mu_1^{-1}w$ in H, hence $\|u_1\| = 1$. It is easy to check that $Ku_1 = w = \mu_1 u_1$, because

$$\|Ku_1 - w\| \le \|Ku_1 - Kv_i\| + \|Kv_i - w\| \le \|K\|\|u_1 - v_i\| + \|Kv_i - w\| \to 0. \tag{28}$$

Now let $H_1 = \{v \in H : \langle v, u_1\rangle = 0\}$, which is a closed linear subspace of H. Moreover, H_1 is invariant under K, because

$$\langle Kv, u_1\rangle = \langle v, Ku_1\rangle = \mu_1\langle v, u_1\rangle = 0, \qquad \text{for } v \in H_1. \tag{29}$$

So if H_1 is nontrivial, we can construct as above an element $u_2 \in H_1$ with $\|u_2\| = 1$ and a number $\mu_2 = \|K\|_{H_1} > 0$ such that $Ku_2 = \mu_2 u_2$. By induction, we have two sequences $\{u_n\} \subset H$ and $\mu_1 \ge \mu_2 \ge \ldots$ satisfying $Ku_n = \mu_n u_n$ and the formula (25). Then the preceding lemma implies that $\mu_n \to 0$ and that $\{u_n\}$ can be chosen to be orthonormal.

To show completeness of $\{u_n\}$, assume that $u \in H$ satisfies $\langle u, u_n\rangle = 0$ for all n. This means that $u \in H_n$ for any n, hence $\|Ku\| \le \mu_n\|u\| \to 0$ as $n \to 0$. Therefore, $Ku = 0$ and this is only possible if $u = 0$ by positivity of K (or by the fact that 0 is not an eigenvalue). □

Corollary 7. *For any $u \in H$, we have*

$$Ku = \sum_n \mu_n\langle u, u_n\rangle u_n, \tag{30}$$

with the convergence in H.

Proof. It is obvious that the map $P_j : H \to \operatorname{span}\{u_1, \ldots, u_j\}$ defined by

$$P_j u = \textstyle\sum_{n=1}^j \langle u, u_n\rangle u_n, \tag{31}$$

is the orthogonal projector onto $\operatorname{span}\{u_1, \ldots, u_j\}$. In particular, we have $w_j = u - P_j u \in H_j$. In view of

$$Kw_j = Ku - \textstyle\sum^j \mu_n\langle u, u_n\rangle u_n, \tag{32}$$

we need to show that $Kw_j \to 0$ as $j \to \infty$. But this follows from

$$\|Kw_j\| \le \mu_{j+1}\|w_j\| \le \mu_{j+1}\|u\|, \tag{33}$$

since H_j is invariant under K and $\mu_{j+1} = \|K\|_{H_j}$. □

3. Application to the Laplace eigenproblems

It is time to apply the general spectral theory to the Laplace eigenvalue problems. We assume that $\Omega \subset \mathbb{R}^n$ is an open set, and $V = H_0^1(\Omega)$ or $V = H^1(\Omega)$. We also assume that for some $t \in \mathbb{R}$, the resolvent $R_t : L^2(\Omega) \to L^2(\Omega)$ is compact and positive. While the positivity assumption is easily satisfied for t large enough, recall that in the Dirichlet case, the compactness assumption is valid if Ω is bounded. In the Neumann case, as we know, one needs some additional requirements on the boundary of the domain Ω.

Since R_t is symmetric, the results from the preceding section imply the existence of an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of R_t. Moreover, each eigenvalue has a finite multiplicity, they are all positive and accumulate at 0. Now the equivalence between the Laplace eigenvalue problem (11) and the resolvent formulation (14) leads us to the following result, which can be regarded as a special case of the *Hodge theorem.*

Theorem 8. *There exists an orthonormal basis $\{u_k\}$ of $L^2(\Omega)$ consisting of eigenfunctions of A. Each eigenvalue has a finite multiplicity, and with $\{\lambda_k\}$ denoting the eigenvalues, we have $\lambda_k > -t$ for all k, and $\lambda_k \to \infty$ as $k \to \infty$. Moreover, the eigenfunctions are real analytic in Ω, and smooth up to the boundary if $\partial\Omega$ is smooth.*

Proof. Let $\{u_k\}$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of R_t, and denote by $\{\mu_k\}$ the corresponding eigenvalues. Then by the equivalence between (11) and (14), the functions $\{u_k\}$ are also eigenfunctions of A, with the eigenvalues $\{\lambda_k\}$ given by

$$\lambda_k = \frac{1}{\mu_k} - t. \tag{34}$$

Positivity of μ_k implies $\lambda_k > -t$, and $\mu_k \to 0$ implies $\lambda_k \to \infty$. The regularity statements are consequences of the regularity theory of Poisson's equation (with a lower order term). □

Remark 9. In the Dirichlet case, since we can take $t = 0$, all eigenvalues are strictly positive. In the Neumann case, we can take any $t < 0$, hence all eigenvalues are nonnegative. The latter estimate is sharp, since $\lambda = 0$ is a Neumann eigenvalue with the eigenfunction $u \equiv 1$.

Lemma 10. *The eigenfunctions $\{u_k\}$ form an orthogonal basis of V with respect to the standard inner product inherited from $H^1(\Omega)$. In particular, for $u \in V$, the expansion*

$$u = \textstyle\sum_k \langle u, u_k\rangle u_k, \tag{35}$$

converges in V.

Proof. Orthogonality of the eigenfunctions with respect to the H^1 inner product follows from

$$\int_\Omega \nabla u_j \cdot \nabla u_k = a(u_j, u_k) = \langle Au_j, u_k\rangle = \lambda_j \delta_{jk}. \tag{36}$$

For $u \in V$, we have

$$a(u_j, u) = \langle Au_j, u\rangle = \lambda_j \langle u_j, u\rangle, \quad \text{so that} \quad \langle u, u_j\rangle_{H^1} = (1 + \lambda_j)\langle u, u_j\rangle. \tag{37}$$

Since $\lambda_j \geq 0$, if $u \in V$ satisfies $\langle u, u_j\rangle_{H^1} = 0$ for all j, then $\langle u, u_j\rangle = 0$ for all j, hence by completeness of the eigenfunctions in $L^2(\Omega)$, we get $u = 0$. This shows the eigenfunctions are complete in V. The convergence of (35) is basic Hilbert space theory: One first shows the convergence of the right hand side by deriving Bessel's inequality, and then uses completeness to infer that the difference between the left and the right hand sides is 0. □

Before closing this section, we want to derive simple variational characterizations of the eigenvalues in terms of the *Rayleigh quotient*

$$\rho(u) \equiv \rho(u, \Omega) = \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}, \qquad u \in H^1(\Omega). \tag{38}$$

In what follows, these characterizations will provide the basic device by which we extract precise spectral information.

Theorem 11. *Suppose that the eigenvalues are ordered so that $\lambda_1 \leq \lambda_2 \leq \ldots$, counting multiplicities. For each k, we have*

$$\lambda_k = \min_{H_{k-1}} \rho, \tag{39}$$

where $H_{k-1} = \{u \in V : \langle u, u_j\rangle = 0,\ j = 1, \ldots, k-1\}$, and if $u \in H_{k-1}$ satisfies $\rho(u) = \lambda_k$ then $Au = \lambda_k u$. We also have

$$\lambda_k = \max_{\mathrm{span}\{u_1,\ldots,u_k\}} \rho, \tag{40}$$

with the maximum attained only by the eigenfunctions corresponding to λ_k.

Proof. For $u \in L^2(\Omega)$, the series $\sum_k \langle u, u_k\rangle u_k$ converges in $L^2(\Omega)$ to u. This in combination with continuity of the L^2 inner product implies *Plancherel's identity*

$$\|u\|^2_{L^2(\Omega)} = \langle u, u\rangle = \langle \textstyle\sum_k \langle u, u_k\rangle u_k, u\rangle = \sum_k |\langle u, u_k\rangle|^2. \tag{41}$$

Similarly, for $u \in V$, by continuity of $a : V \times V \to \mathbb{R}$ we have

$$\|\nabla u\|^2_{L^2(\Omega)} = a(u, u) = a(\textstyle\sum_k \langle u, u_k\rangle u_k, u) = \sum_k \langle u, u_k\rangle a(u_k, u) = \sum_k \lambda_k |\langle u, u_k\rangle|^2, \tag{42}$$

where we have used $a(u_k, u) = \lambda_k \langle u_k, u\rangle$ and the fact that $\lambda_k \geq 0$. If in addition, $u \in H_{j-1}$, i.e., if $u \perp_{L^2} \operatorname{span}\{u_1, \ldots, u_{j-1}\}$, then

$$\|\nabla u\|^2_{L^2(\Omega)} = \sum_{k\geq j} \lambda_k |\langle u, u_k\rangle|^2 \geq \lambda_j \sum_{k\geq j} |\langle u, u_k\rangle|^2 = \lambda_j \|u\|^2_{L^2(\Omega)}, \tag{43}$$

with the equality occurring if and only if u is in the eigenspace of λ_j. We have established (39), and the fact that $u \in H_{j-1}$ satisfies $\rho(u) = \lambda_j$ if and only if $Au = \lambda_j u$.

The characterization (40) follows from the fact that for $u \in \operatorname{span}\{u_1, \ldots, u_j\}$, we have

$$\|\nabla u\|^2_{L^2(\Omega)} = \sum_{k\leq j} \lambda_k |\langle u, u_k\rangle|^2 \leq \lambda_j \sum_{k\leq j} |\langle u, u_k\rangle|^2 = \lambda_j \|u\|^2_{L^2(\Omega)}, \tag{44}$$

with the equality occurring if and only if u is in the eigenspace of λ_j. □

Remark 12. We deduce from the previous theorem that the best constant c in the inequality

$$\|u\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega)} \qquad \text{for all} \quad u \in H_{k-1}, \tag{45}$$

is $c = \lambda_k^{-\frac{1}{2}}$. Note that when $V = H^1_0(\Omega)$ and $k = 1$ the inequality reduces to the Friedrichs inequality, and when $V = H^1(\Omega)$ and $k = 2$ it reduces to the Poincaré inequality. Hence the first Dirichlet eigenvalue and respectively the second Neumann eigenvalue characterize the sharp constants in those classical inequalities.

Exercise 13. Compute the sharp constant of the Poincaré inequality for a rectangle.

Exercise 14. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set having the H^1 extension property. Show that the dimension of the eigenspace corresponding to the first Neumann eigenvalue (i.e., the multiplicity of λ_1) is equal to the number of connected components of Ω.

4. Eigenvalue comparison theorems

Note that the variational characterizations

$$\lambda_k = \max_{u\in\operatorname{span}\{u_1,\ldots,u_k\}} \rho(u) = \min_{u\in H_{k-1}} \rho(u), \tag{46}$$

given by Theorem 11 involve the eigenfunctions $\{u_j\}$, so it is not very convenient if, e.g., one is only interested in the eigenvalues. In any case, it is possible to remove the dependence on eigenfunctions altogether by adding one more layer of extremalization, because the space $\operatorname{span}\{u_1, \ldots, u_k\}$ is positioned in an optimal way inside the manifold of k dimensional subspaces of V (this manifold is called the k-th Grassmannian of V).

Theorem 15 (Courant's minimax principle)**.** *We have*

$$\lambda_k = \min_{X\in\Phi_k} \max_X \rho, \tag{47}$$

where $\Phi_k = \Phi_k(V) = \{X \subset V \text{ linear subspace} : \dim X = k\}$.

Proof. The first equality in (46) shows that

$$\lambda_k \geq \min_{X\in\Phi_k} \max_{u\in X} \rho(u). \tag{48}$$

On the other hand, if $X \in \Phi_k$, then $X \cap H_{k-1}$ is nontrivial by dimensional considerations. This means that there is a nonzero $v \in X \cap H_{k-1}$, hence $\rho(v) \geq \lambda_k$ by the second characterization in (46), that is, $\max_X \rho \geq \lambda_k$. As $X \in \Phi_k$ was arbitrary, we conclude

$$\lambda_k \leq \min_{X\in\Phi_k} \max_{u\in X} \rho(u), \tag{49}$$

establishing the theorem. □

Since $H_0^1(\Omega)$ is a subspace of $H^1(\Omega)$, it is clear that $\Phi_k(H_0^1(\Omega)) \subset \Phi_k(H^1(\Omega))$. This observation leads to the following simple inequality between Dirichlet and Neumann eigenvalues.

Corollary 16. *Supposing that Ω is a bounded domain having the H^1 extension property, let us denote by $\mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots$ the Dirichlet eigenvalues of Ω, and by $\nu_1(\Omega) \leq \nu_2(\Omega) \leq \ldots$ the Neumann eigenvalues of Ω. Then we have $\nu_k(\Omega) \leq \mu_k(\Omega)$ for all k.*

Remark 17. In 1991, Leonid Friedlander proved that $\nu_{k+1} \leq \mu_k$.

The next corollary is based on the observation that if $\Omega_1 \subset \Omega_2$ then $H_0^1(\Omega_1) \subset H_0^1(\Omega_2)$.

Corollary 18 (Domain monotonicity)**.** *If $\Omega_1 \subset \Omega_2$ are bounded domains, then we have $\mu_k(\Omega_2) \leq \mu_k(\Omega_1)$ for all k.*

Proof. For $u \in H_0^1(\Omega_1)$, let us denote by $\tilde{u} \in L^2(\Omega_2)$ the extension of u by 0 outside Ω_1. We claim that $\tilde{u} \in H_0^1(\Omega_2)$ with $\|\tilde{u}\|_{H^1(\Omega_2)} = \|u\|_{H^1(\Omega_1)}$. If the claim is true, $H_0^1(\Omega_1)$ can be considered as a subspace of $H_0^1(\Omega_2)$, and

$$\rho(u, \Omega_2) = \frac{\|\nabla u\|^2_{L^2(\Omega_2)}}{\|u\|^2_{L^2(\Omega_2)}} = \frac{\|\nabla u\|^2_{L^2(\Omega_1)}}{\|u\|^2_{L^2(\Omega_1)}} = \rho(u, \Omega_1), \tag{50}$$

for $u \in H_0^1(\Omega_1)$, where extension of u by 0 outside Ω_1 is understood in necessary places.

To see that the claim is true, let $\{\phi_k\} \subset \mathscr{D}(\Omega_1)$ be a sequence converging to u in $H^1(\Omega_1)$. Passing to a subsequence if necessary, we can arrange that ϕ_k converges almost everywhere in Ω_1 to u. Then we extend each ϕ_k by 0 outside Ω_1, and note that ϕ_k converges almost everywhere in Ω_2 to $\tilde{u}$. Now the equality $\|\phi_j - \phi_k\|_{H^1(\Omega_1)} = \|\phi_j - \phi_k\|_{H^1(\Omega_2)}$ and the completeness of $H_0^1(\Omega_2)$ imply that ϕ_k converges in the $H^1(\Omega_2)$ norm to some $v \in H_0^1(\Omega_2)$. Again passing to a subsequence if necessary, the convergence is almost everywhere in Ω_2. Hence $v = \tilde{u}$ almost everywhere in Ω_2, which means that $\tilde{u} \in H_0^1(\Omega_2)$. □

As it turns out, domain monotonicity does not hold for Neumann eigenvalues.

Example 19. The Neumann eigenvalues of the rectangle with sides a and b are

$$\nu_{k,\ell} = \frac{(\pi k)^2}{a^2} + \frac{(\pi \ell)^2}{b^2}, \qquad k, \ell \in \mathbb{N}_0. \tag{51}$$

So assuming that $a > b$, the first 3 eigenvalues are

$$\nu_1 = 0, \qquad \nu_2 = \frac{\pi^2}{a^2}, \qquad \text{and} \qquad \nu_3 = \frac{\pi^2}{b^2}. \tag{52}$$

We pick $1 < a < \sqrt{2}$, and choose $b > 0$ small, so that the rectangle can be placed inside the unit square. For the unit square, the first 3 Neumann eigenvalues are

$$\nu_1' = 0, \qquad \nu_2' = \pi^2, \qquad \text{and} \qquad \nu_3' = \pi^2. \tag{53}$$

Since $a > 1$, we have $\nu_2 < \nu_2'$, which could not happen if domain monotonicity were true. ⊘

Even though domain monotonicity is not true in the strict sense, a weakened form of domain monotonicity holds for the Neumann eigenvalues.

Corollary 20 (Weak domain monotonicity)**.** *Suppose that* Ω_1 *is a bounded domain having the* H^1 *extension property, and let* Ω_2 *be another bounded domain such that* $\bar{\Omega}_1 \subset \Omega_2$*. Then there exists a constant* c *such that* $\nu_k(\Omega_2) \leq \mu_k(\Omega_2) \leq c\nu_k(\Omega_1)$ *for all* k*.*

With the extension operator $E : H^1(\Omega_1) \to H^1_0(\Omega_2)$ playing the role of an injection, the proof of the preceding corollary is similar to that of Corollary 18.

Exercise 21 (Maximin principle)*.* Show that

$$\lambda_k = \max_{X \in \Phi_{k-1}} \inf_{u \in X^\perp} \rho(u), \tag{54}$$

where $X^\perp$ is understood as $\{u \in V : u \perp_{L^2} X\}$.